A UNIFIED STUDY OF CERTAIN SUBCLASSES OF UNIVALENT ANALYTIC FUNCTIONS

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To my parents

and wife

12/2/76

CERTIFICATE

This is to certify that the work embodied in the thesis "A unified study of certain subclasses of univalent analytic functions" by Manak Lal Mogra has been carried out under my supervision and has not been submitted elsewhere for a degree or diploma.

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(Manak Lal Mogra)

CONTENTS

SYNOPSIS

From the point of view of conformal mappings, the class of univalent analytic functions is the simplest but most important class of analytic functions. The failure to settle Bieberbach's conjecture in its generality has led to the investigation of several subclasses of univalent analytic functions. Amongst these subclasses, the class of starlike functions plays such a vital role in the study of univalent functions, that, not only has this class been extensively studied, but many of its subclasses have been introduced and studied by different workers separately (see e.g. Robertson (1936), MacGregor (1963), Libera (1964), Padmanabhan (1968), Wright (1969), Eenigenburg (1972), McCarty (1974) etc.).

In the present thesis, a unified approach to the study of various subclasses of starlike functions has been made by introducing the class $S^*(\alpha,\beta)$ of starlike functions of order $\alpha(0 \leq \alpha < 1)$ and type $\beta(0 < \beta \leq 1)$. This, for different values of the parameters α and β not only leads to the classes introduced by the above mentioned workers but also gives rise to many new subclasses of starlike functions. A similar approach has also been provided for a unified study of various other subclasses of univalent functions.

have been investigated in the remaining five chapters, also find a mention here.

In chapter II, we obtain a representation formula, coefficient estimates, distortion theorems etc. for the class $S^*(\alpha,\beta)$ of starlike functions of order α and type β. A sufficient condition for a function to be in $S^*(\alpha,\beta)$ has also been obtained. For different values of the parameters α,β our results yield the results obtained by MacGregor (Michigan Math. J. 10 (1963), 277-281), Schild (Amer. J. Math. 87 (1965), 65-70), Wright (Compositio Math. 21 (1969), 122-124), McCarty (Proc. Amer. Math. Soc. 43 (2) (1974), 361-366) and others.

Chapter III deals with the determination of the radii of convexity for starlike functions of order $\alpha (0 \leq \alpha < 1)$ and type $\beta (0 < \beta \leq 1)$. In particular, our results include the results obtained by MacGregor (Proc. Amer. Math. Soc. 14 (1963), 71-76), Zmorovic (Amer. Math. Soc. Transl. (2) 80 (1969), 203-213), Singh and Goel (J. Math. Soc. Japan 23 (1971), 323-339), Eenigenburg (Compositio Math. 24 (2) (1972), 235-238), McCarty (Proc. Amer. Math. Soc. 43 (2) (1974), 361-366) etc. We also determine the radii of starlikeness for functions of the form $2F(z) = (zf(z))'$ where $f \in S^*(\alpha,\beta)$. The results due to Livingston (Proc. Amer. Math. Soc. 17 (1966), 352-357), Singh and Goel (J. Math. Soc. Japan 23 (1971), 323-339), Libera and Livingston (Proc. Amer. Math. Soc. 30 (2) (1971), 361-370), Al-Amiri (Proc. Amer. Math. Soc. 39 (1) (1973), 101-109) etc. follow from our results.

In the IVth chapter, we introduce a new class $R(\alpha,\beta)$ of functions whose derivatives have a positive real part in the unit disc and obtain coefficient estimates, distortion theorems, radius of convexity etc. for functions in this class. These results sharpen and generalize the various results obtained earlier by MacGregor (Trans. Amer. Math. Soc. 104 (1962), 532-537; Proc. Amer. Math. Soc. 14 (1963), 514-524), Caplinger and Causey (Proc. Amer. Math. Soc. 39 (2) (1973), 357-361), Shaffer (Proc. Amer. Math. Soc. 39 (2) (1973), 281-287) and others.

Chapter V is devoted to determine radii of starlikeness and convexity for several subclasses of univalent functions. The results obtained in this chapter generalize and include the results determined by Livingston (Proc. Amer. Math. Soc. 17 (1966), 352-357), Ratti (Math. Z. 107 (1968), 241-248), Singh and Goel (J. Math. Soc. Japan 23 (1971), 323-339), Nikolaeva and Repnina (Ukrain Mat. Z. 24 (1972), 268-273) etc. A different technique has been adopted for obtaining the radii of convexity and starlikeness for deriving the results obtained earlier by MacGregor (Proc. Amer.Math. Soc. 14 (1963), 71-76, Trans. Amer. Math. Soc. 104 (1962), 532-537), Ratti (Math. Z. 107 (1968), 241-248), Hengartner and Schober (Proc. Amer. Math. Soc. 28 (1971), 519-524) and others.

In the last chapter, for the various subclasses introduced in the preceding chapters, we first obtain distortion theorems when the second coefficient in the power series expansion of the function in

question is kept fixed. We then investigate the effect of second coefficient on the radii of convexity and starlikeness for the functions of the several subclasses of univalent functions. These results yield the results obtained earlier by Tepper (Trans. Amer. Math. Soc. 150 (1970), 519-528), Gupta R.S. (J. Aust. Math. Soc. 14 (1) (1972), 1-8), McCarty (Proc. Amer. Math. Soc. 35 (1972), 211-216; 42 (1974), 153-160), Al-Amiri (Proc. Amer. Math. Soc. 42 (1974), 466-474) and others.

LIST OF SYMBOLS USED FOR THE VARIOUS SUBCLASSES

Symbol	Introduced on page	Symbol	Introduced on page
$\mathcal{A}$	18	$\mathcal{Q}_k(\alpha,\delta,\lambda)$	97
$\mathcal{B}$	45	$\bar{\mathcal{Q}}_k(\gamma,\delta,\lambda)$	97
$\mathcal{C}$	3	$\mathcal{R}$	8
$\mathcal{C}_\alpha$	3	$\bar{\mathcal{R}}$	8
$\mathcal{C}(\alpha,\beta)$	17	$\mathcal{R}_\alpha$	8
$\mathcal{C}_{\alpha,k}$	97	$\mathcal{R}(\gamma)$	9
$\mathcal{C}_k(\gamma)$	97	$\bar{\mathcal{R}}(\delta)$	8
$\mathcal{C}_k(\alpha,\beta)$	22	$\bar{\mathcal{R}}^*(\alpha)$	8
$\mathcal{D}_k^*(\alpha,\beta)$	96	$\bar{\mathcal{R}}^{**}(\delta)$	9
$\mathcal{E}_k(\alpha,\beta)$	96	$R(\alpha,\beta)$	73
$\mathcal{F}(\alpha,\beta,\lambda)$	96	R_k	10
$\mathcal{G}^*(\alpha,\beta,\lambda)$	96	$\bar{R}_k$	10
$\bar{\mathcal{G}}^*(\alpha,\beta,\lambda)$	97	$R_{\alpha,k}$	10
$\mathcal{H}$	114	$R_k(\gamma)$	10
$\mathcal{H}^*(\gamma,\delta)$	115	$\bar{R}_k(\delta)$	10
$\mathcal{H}_k^*(\alpha,1,\lambda)$	115	$\bar{R}_k^*(\alpha)$	10
$\mathcal{H}_k^*(\alpha,\delta,\lambda)$	97	$\bar{R}_k^{**}(\delta)$	10
$\bar{\mathcal{H}}_k^*(\gamma,\delta,\lambda)$	97	$R_k(\alpha,\beta)$	74
$\mathcal{K}$	6	$R(a;\alpha)$	134
$\mathcal{P}$	7	$\bar{R}(a;\alpha)$	134
$\mathcal{P}(\alpha,\beta)$	123	$R(a;\alpha,\beta)$	124
$\mathcal{P}(b;\alpha,\beta)$	124	S	1

Symbol	Introduced on page	Symbol	Introduced on page
S^*	4	$\bar{S}^*_k(1-\alpha)$	10
$\bar{S}^*$	58	$S^*_k(\alpha,\beta)$	18
S^*_α	4	$\bar{S}^{**}_a$	137
$S^*(\gamma)$	5	$S^*(a;\alpha)$	137
$\bar{S}\ (\delta)$	5	$S^*(a;\alpha,\beta)$	124
$\bar{S}^*(\alpha)$	6	$\bar{V}^*$	66
$\bar{S}^*(1-\alpha)$	5	V^*_α	66
S^γ	7	$\bar{V}(\delta)$	66
$S^*(\alpha,\beta)$	16	$V^*(\gamma)$	66
S_k	9	$\bar{V}^*(\alpha)$	66
S^*_k	40	$\bar{V}^*(1-\alpha)$	66
$\bar{S}^*_k$	22	$V^*(\alpha,\beta)$	44
$S^*_{\alpha,k}$	10	$\bar{V}^*_a$	140
$S^*{}_k(\gamma)$	10	$V^*(a;\alpha,\beta)$	124
$\bar{S}_k(\delta)$	10	$\sum$	96
$\bar{S}^*_k(\alpha)$	10		

CHAPTER I

INTRODUCTION

1.1 A function g is said to be univalent (schlicht, simple) in a domain E, if it takes no value more than once in E. Equivalently, if $g(z_1) = g(z_2)$ with z_1, z_2 in E, then $z_1 = z_2$. A necessary condition for an analytic function g to be univalent in a domain E is that $g'(z) \neq 0$ for every $z \in E$. Thus, if the domain E under consideration is the unit disc $\Delta \equiv \{z: |z| < 1\}$, then $g(z)$ is analytic and univalent in Δ if, and only if, the function $f(z) = \{g(z) - g(0)\}/g'(0)$ is analytic and univalent in Δ. We shall denote by S the class of all functions that are analytic and univalent in Δ and which satisfy the conditions $f(0) = 0$, $f'(0) = 1$. Every such function has a power series representation about origin as

$$(1.1.1) \qquad f(z) = z + \sum_{n=2}^{\infty} a_n z^n .$$

The beginning of the theory of univalent analytic functions (henceforth called univalent functions) may be assigned to a paper by Koebe, who, in 1907, showed, alongwith other results, that there exists a constant K (called Koebe's constant) such that the distance of the boundary of $f(\Delta)$ for any $f \in S$ is not less

than K from $w = 0$. This paper soon attracted the attention of other eminent mathematicians like Bieberbach, Gronwall, Faber and others. In particular, Bieberbach [7] showed that the exact value of Koebe's constant comes out to be 1/4. He also obtained precise bounds for $|f'(z)|$, $|f(z)|$ for $f \in S$. With the help of Gronwall's area principle, Bieberbach also showed that $|a_2| \leq 2$ for every $f \in S$. Since, with regard to all these properties, the function $z(1+\varepsilon z)^{-2}$, $|\varepsilon| = 1$, turns out to be an extremal function and since for this function $|a_n| = n$ for $n \geq 2$, this made Bieberbach conjecture that, in general $|a_n| \leq n$ for $n \geq 2$ for all $f \in S$. This conjecture has been shown to be true for $n = 3,4,5,6$ by Löwner [40], Garabedian and Schiffer [17], Pederson and Schiffer [66] and Pederson [65] respectively. Recently, Ozawa and Kubota [58] have proved that $\text{Re } a_8 \leq 8$ if $1.9 \leq \text{Re } a_2 \leq 2$ and $|\text{Im } a_2/\text{Re } a_2| \leq 1/20$. Validity of this conjecture for a_7 and $a_n (n \geq 9)$ still remains an open problem.

Attempts to find a global proof for Bieberbach's conjecture have met with a partial success. Bieberbach [7] himself showed that $|a_n| < 5.1\, n^2$ for all n. In 1925, Littlewood [38] improved this estimate to $|a_n| < en$. In 1951, Bazilevič [5] succeeded in showing that $|a_n| < 1/2\, en + 1.51$. While Hayman [24] showed that $|a_n| \leq n$ for $n > n_o(f)$, Fitzgerald [16] in 1972, obtained that

$|a_n| < (1.0801)n$. Recently, Horowitz [26] has claimed to have proved the estimate $|a_n| < (1.0691)n$ for $n \geq 2$.

1.2 Failure to establish the Bieberbach conjecture in its generality has given rise to the investigation of various subclasses of S. Foremost amongst these is the class of convex functions.

A function $f \in S$ is said to be convex in Δ if image of Δ under f is a convex region, i.e., the line segment joining any two points of $f(\Delta)$ lies wholly in $f(\Delta)$. We shall denote by C, the class of all functions of S which are convex in Δ.

A necessary and sufficient condition for a function $f \in S$ to be convex has been given by Robertson [71]. It states that a function $f \in S$ is convex in Δ if, and only if,

$$(1.2.1) \qquad \text{Re}\ \{1 + z\frac{f''(z)}{f'(z)}\} > 0,\ z \in \Delta .$$

Robertson [71] also introduced the concept of order for functions in C. A function $f \in C$ is of order $\alpha (0 \leq \alpha < 1)$ if

$$(1.2.2) \qquad \text{Re}\ \{1 + z\frac{f''(z)}{f'(z)}\} > \alpha \text{ for } z \in \Delta .$$

The class of convex functions of order α, we shall denote by C_α.

1.3 Another subclass of S which is wider than C, is the class of starlike functions.

A function $f(z)$ in S is said to be starlike in Δ if the image of Δ under f containing the point $w = 0$ is a starlike region, i.e., the line segment joining $w = 0$ to any point of $f(\Delta)$ lie wholly in $f(\Delta)$. We shall denote by S^* the class of all functions in S which are starlike w.r.t. origin.

A necessary and sufficient condition for $f \in S$ to be starlike is also due to Robertson [71]. Thus, a function $f \in S$ is starlike in Δ w.r.t. origin if, and only if,

$$(1.3.1) \qquad \mathrm{Re}\ \{z \frac{f'(z)}{f(z)}\} > 0,\ z \in \Delta .$$

The concept of order for a starlike function was initially introduced by Robertson [71]. A function $f \in S$ is starlike of order $\alpha(0 \leq \alpha < 1)$ in Δ if

$$(1.3.2) \qquad \mathrm{Re}\ \{z \frac{f'(z)}{f(z)}\} > \alpha$$

for all $z \in \Delta$. We shall denote by S^*_α, the class of starlike functions of order $\alpha(0 \leq \alpha < 1)$ in Δ. It is clear from (1.2.2) and (1.3.2) that

$$(1.3.3) \qquad f \in C_\alpha \text{ if, and only if, } zf' \in S^*_\alpha .$$

Further, it is to be noted that $C \subset S^*$. In fact, Marx [50] and Strohhacker [80] proved that every convex function is starlike

of order 1/2 and that this result is sharp. Later, Jack [27] proposed a general problem, viz., if $f \in C_\alpha$, find $\beta(\alpha)$ such that $f \in S^*_{(\beta(\alpha))}$. He also obtained a partial solution to this problem. Recently, this problem has been completely solved by MacGregor [48].

The class of starlike functions plays such a vital role in the study of univalent functions that various subclasses of this class have been introduced and studied by different workers separately. Here we give a brief account of some of them.

Ram Singh [69] introduced the class $\bar{S}(\delta)$ of functions $f \in S$ which satisfy

(1.3.4) $$\left|z\frac{f'(z)}{f(z)} - \delta\right| < \delta, \; 1/2 < \delta \le 1, \; z \in \Delta.$$

Later, Padmanabhan [61] introduced $S^*(\gamma)$ as a subclass of S^*. Thus, according to him, $f \in S^*(\gamma)$ if, and only if,

(1.3.5) $$\left|\left(z\frac{f'(z)}{f(z)} - 1\right) / \left(z\frac{f'(z)}{f(z)} + 1\right)\right| < \gamma$$

for some $\gamma (0 < \gamma \le 1)$ and for all $z \in \Delta$.

Yet another subclass $\bar{S}^*(1-\alpha)$ of starlike functions has been introduced by Wright [83]. Thus, $f \in \bar{S}^*(1-\alpha)$ if, and only if,

(1.3.6) $$\left|z\frac{f'(z)}{f(z)} - 1\right| < 1 - \alpha$$

holds for some $\alpha (0 \le \alpha < 1)$ and $z \in \Delta$.

Eenigenburg [13] considered the class $\bar{S}^*(\alpha)$ of functions f in S satisfying

$$(1.3.7) \qquad \left|z \frac{f'(z)}{f(z)} - 1\right| < \alpha, \ 0 < \alpha \leq 1, \ z \in \Delta .$$

1.4 Another class wider than the class of starlike functions is the class of close-to-convex functions introduced by Kaplan [32]. If $f \in S$, then f is close-to-convex if there exists a function $\phi \in C$ such that

$$(1.4.1) \qquad \text{Re } \{f'(z)/\phi'(z)\} > 0, \ z \in \Delta .$$

This class we denote by K. Kaplan [32] has also characterised close-to-convex functions, without reference to a convex function. Thus, $f \in S$ is close-to-convex if, and only if,

$$(1.4.2) \qquad \int_{\theta_1}^{\theta_2} \text{Re } \{1 + z \frac{f''(z)}{f'(z)}\} \, d\theta > -\pi$$

where $\theta_1 < \theta_2$, $z = re^{i\theta}$ and $r < 1$.

Libera [34] showed that if $f \in C$, S^* or K then $F(z) = (2/z) \int_0^z f(t)\, dt$ also belongs to C, S^* or K respectively. Livingston [39] studied the converse problem of this. He proved that if $f \in C, S^*$ or K then $F(z) = 1/2\ (zf(z))'$ belongs to these classes respectively for $|z| < 1/2$. Later, Bernardi [6] generalized these results to functions of the form $F(z) = ((c + 1)/z^c) \int_0^z t^{c-1} f(t)dt$ where $c = 1,2\ldots.$. Further

work in this direction has been done by Libera and Livingston [37] Singh and Goel [78] , Al-Amiri [2] and others.

Yet another class wider than the class of starlike functions is the class of spiral like functions introduced by L-Spaček [79] In 1932, Spaček essentially showed that a function f satisfying

$$(1.4.3) \qquad \text{Re } \{\xi \frac{zf'(z)}{f(z)}\} > 0, \ |\xi| = 1, \quad z \in \Delta$$

is univalent in Δ. If we replace ξ by $e^{i\gamma}$, $|\gamma| < \pi/2$, then f is called a γ-spiral function. The class of γ-spiral functions, which we denote by S^{γ}, was studied by Libera [36]. Obviously $S^{0} \equiv S^{*}$.

Some analogous extensions of the classes S^{*}_{α}, C_{α}, K etc. have also been carried over to the meromorphic univalent functions which are regular in the unit disc except for a simple pole at $z = 0$ (see [31], [35], [47], [67] and others).

1.5 Let P denote the class of functions $p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ analytic in Δ and satisfying

$$(1.5.1) \qquad \text{Re } (p(z)) > 0$$

for $z \in \Delta$. It is well known ([57], [82] etc.) that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in a convex domain E and satisfies

(1.5.2) $\quad \text{Re}\,(f'(z)) > 0$

for all $z \in E$, then f is univalent in E. MacGregor [41] has studied the class R of functions $f \in S$ such that $f' \in P$. The class R_α of functions, of the form (1.1.1), analytic in Δ, and satisfying

(1.5.3) $\quad \text{Re}\,(f'(z)) > \alpha$

for $0 \leq \alpha < 1$ and $z \in \Delta$, has been studied by Ezrohi [14] Martynov [49] etc.

Several other subclasses of analytic functions whose derivatives have a positive real part in the unit disc have also been introduced and studied.

MacGregor [46] introduced and studied the subclass of R denoted by $\bar{R}$ and defined by the condition

(1.5.4) $\quad |f'(z) - 1| < 1, \; z \in \Delta$.

Goel [18] generalized the above class by studying the class $\bar{R}(\delta) \subset R$ satisfying the condition

(1.5.5) $\quad |f'(z) - \delta| < \delta, \; 1/2 < \delta \leq 1, \; z \in \Delta$.

Goel [19] also introduced and studied the class $\bar{R}^*(\alpha)$ of functions $f \in S$ satisfying

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$$(1.5.6) \qquad |f'(z) - 1| < \alpha$$

for some $\alpha (0 < \alpha \leq 1)$ and all $z \in \Delta$.

Another interesting subclass $\mathcal{R}(\gamma)$ of $\mathcal{R}$ has been introduced and studied by Padmanabhan [64] and Caplinger and Causey [10]. Thus, according to them, a function $f \in \mathcal{R}(\gamma)$ if, and only if,

$$(1.5.7) \qquad |(f'(z)-1)/(f'(z) + 1)| < \gamma$$

for some $\gamma(0 < \gamma \leq 1)$ and for all $z \in \Delta$. Recently Shaffer [76] has introduced the class $\bar{R}^{**}(\delta)$ of functions $f(z)$ of the form (1.1.1) and satisfying the condition

$$(1.5.8) \qquad \left|f'(z) - \frac{1}{2\delta}\right| < \frac{1}{2\delta}$$

for $0 \leq \delta < 1$ and $z \in \Delta$.

1.6 Recently, the study of some of the above subclasses of univalent functions has been extended by considering the power series of $f \in S$ to be of the form

$$(1.6.1) \qquad z + \sum_{n=k+1}^{\infty} a_n z^n .$$

If we denote the class of functions $f \in S$ having representation (1.6.1) by S_k , then it is clear that $S_1 \equiv S$ and that $S_k \subset S_{k'}$ if $k \geq k'$.

The corresponding subclasses of starlike functions $f \in S_k$ satisfying respectively the conditions (1.3.2), (1.3.4), (1.3.5), (1.3.6) and (1.3.7) we shall denote by $S^*_{\alpha,k}$, $\bar{S}_k(\delta)$, $S^*_k(\gamma)$, $\bar{S}^*_k(1-\alpha)$ and $\bar{S}^*_k(\alpha)$. Analogously, the subclasses of R consisting of functions $f \in S_k$ and satisfying respectively the conditions (1.5.2) to (1.5.8) will be denoted by

R_k, $R_{\alpha,k}$, $\bar{R}_k$, $\bar{R}_k(\delta)$, $\bar{R}^*_k(\alpha)$, $R_k(\gamma)$ and $\bar{R}^{**}_k(\delta)$.

1.7 The theory of univalent functions has been vastly enriched by the introduction of the concept of extreme points, variational methods, integral and parametric representations, methods of symmetrization etc. Here we have described only those aspects of the theory of univalent functions in the direction of which we have pursued it further. Thus, in the present work, an attempt has been made to have a unified study of various subclasses of univalent functions. It will be seen that our approach not only yields a generalization of the various known results but also gives rise to a number of new and refined results. The results are presented in the next five chapters.

In Chapter II, a unified approach to the study of various subclasses of starlike functions has been made by introducing the class $S^*(\alpha,\beta)$ of starlike functions of order $\alpha(0 \leq \alpha < 1)$ and type $\beta(0 < \beta \leq 1)$. Thus, we obtain a representation formula, distortion theorems, coefficient estimates etc. for the class $S^*_k(\alpha,\beta)$ of analytic functions whose power series begins $f(z) = z+a_{k+1}z^{k+1} + \dots$

and which are starlike of order $\alpha(0 \leq \alpha < 1)$ and type $\beta(0 < \beta \leq 1)$. A sufficient condition for a function to be in $S_k^*(\alpha,\beta)$ has also been obtained. For different values of the parameters α and β, our results yield, alongwith some new results, the corresponding results obtained by MacGregor [42, 43], Schild [73, 74], Boyd [8], Ram Singh [68, 69], Padmanabhan [61], Wright [83], Eenigenburg [13], McCarty [53] etc.

In Chapter III, we first prove a lemma which is widely used, throughout this work, in determining the radii of convexity and starlikeness for several subclasses of univalent functions. We then obtain sharp estimates for the radii of convexity for starlike functions of order $\alpha(0 \leq \alpha < 1)$ and type $\beta(0 < \beta \leq 1)$. In particular, these results include the results obtained by MacGregor [43], Zmorovič [84], Singh and Goel [78], Eenigenburg [13], McCarty [53] and others. Moreover, we also obtain the radii of starlikeness for functions of the form $2F(z) = (zf(z))'$ where $f \in S^*(\alpha,\beta)$. The results due to Livingston [39], Singh and Goel [78], Libera and Livingston [37], Al-Amiri [2] etc. follow from our results.

Chapter IV is devoted to a unified study of various subclasses of the class of functions whose derivatives have a positive real part in the unit disc. We introduce a new subclass $R(\alpha,\beta)$ which for different values of the parameters α, β gives the subclasses defined in sec. 1.5. We determine coefficient estimates, distortion theorems

radii of convexity etc. for functions in this class. The results so obtained sharpen and generalize the various results obtained earlier by MacGregor [41 , 46], Padmanabhan [64], Goel[18 ,19] , Caplinger and Causey [10] , Shaffer[75,76] and others. We also determine the radii of convexity for functions in $R_k(\alpha, \beta)$ which include the corresponding results obtained by MacGregor [41], Shaffer [75] etc.

Chapter V deals with the determination of the radii of starlikeness and convexity for several subclasses of univalent functions. These results sharpen and generalize the results obtained by Livingston [39], Shah [77], Padmanabhan [60], Nikolaeva and Repnina [56] etc. We also give a different technique for obtaining the radii of convexity and starlikeness for functions of the classes studied earlier by MacGregor [44,45], Ratti [70], Hengartner and Schober [25] and others.

In the last chapter, we first obtain distortion theorems for various subclasses introduced in the preceding chapters when the second coefficient in the power series expansion of the function in question is kept fixed. We then study the effect of second coefficient on the radii of convexity and starlikeness for the functions of the several subclasses of univalent functions. The results obtained earlier by Tepper [81] , Goel [20], McCarty[51,52], Al-Amiri [1] etc. follow from our results.

CHAPTER II

STARLIKE FUNCTIONS OF ORDER α AND TYPE β-I

2.1 Let $f(z)$ given by

$$(2.1.1) \qquad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be analytic in the unit disc $\Delta \equiv \{z: |z| < 1\}$. If $f(z)$ satisfies the condition

$$(2.1.2) \qquad \operatorname{Re}\left\{z \frac{f'(z)}{f(z)}\right\} > 0$$

for all $z \in \Delta$, then it is well known ([55], pp. 221) that (2.1.2) is both necessary and sufficient for f to be univalent and starlike in Δ. The function $f(z)$, given by (2.1.1), is starlike of order $\alpha (0 \leq \alpha < 1)$ in the unit disc Δ if

$$(2.1.3) \qquad \operatorname{Re}\left\{z \frac{f'(z)}{f(z)}\right\} > \alpha$$

for all $z \in \Delta$ (see [33] , [74] and others). The condition (2.1.3) is equivalent to

$$\left|\left(\frac{zf'(z)}{f(z)} - 1\right) \Big/ \left(z \frac{f'(z)}{f(z)} + 1-2\alpha\right)\right| < 1$$

or

$$\left|\left(z\frac{f'(z)}{f(z)}-1\right)/\left\{2\left(z\frac{f'(z)}{f(z)}-\alpha\right)-\left(z\frac{f'(z)}{f(z)}-1\right)\right\}\right|<1$$

i.e.

$$(2.1.4)\quad \left|z\frac{f'(z)}{f(z)}-1\right| < \left|2\left(z\frac{f'(z)}{f(z)}-\alpha\right)-\left(z\frac{f'(z)}{f(z)}-1\right)\right| .$$

Now, let $w = zf'(z)/f(z)$ and denote the half-plane Re $w > \alpha$ by Π_α. Let A,B denote respectively the points $(\alpha,0)$ and $(1,0)$ in the w-plane. Corresponding to a point P in the w-plane choose P_1 such that $\vec{AP_1} = 2\vec{AP}$ and draw the vector $\vec{P_1B_1}$ equal to the vector $\vec{PB}$.

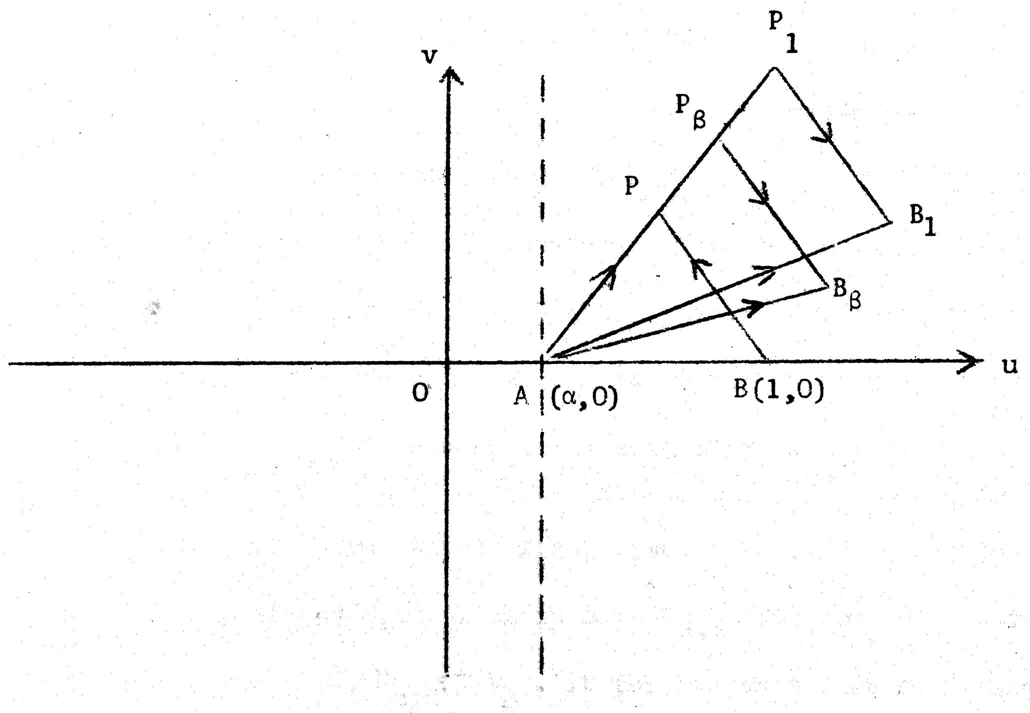

FIG-1

Then the condition for P to be in Π_α is equivalent to saying that

$$(2.1.5) \qquad |\vec{BP}| < |2\vec{AP} - \vec{BP}| = |\vec{AP_1} - \vec{B_1P_1}| = |\vec{AB_1}| .$$

Thus, we find that P lies in Π_α if, and only if, (2.1.5) holds. Now for a given $\beta(0 < \beta \leq 1)$, choose a point P_β such that $\vec{AP_\beta} = 2\beta\ (\vec{AP})$ and draw vector $\vec{P_\beta B_\beta}$ equal to the vector $\vec{PB}$. Consider the locus of the point P satisfying

$$(2.1.6) \qquad |\vec{BP}| < |2\beta\ \vec{AP} - \vec{BP}| = |\vec{AP_\beta} - \vec{B_\beta P_\beta}| = |\vec{AB_\beta}|.$$

It can be easily seen that the locus of P, in this case, will be a subregion $\Pi_{\alpha,\beta}$ of the plane Π_α . In fact, simple calculations show that this subregion $\Pi_{\alpha,\beta}$ is a disc which includes the point B in its interior and has centre at $(1+\alpha-2\alpha\beta)/2(1-\beta)$ and radius $(1-\alpha)/2(1-\beta)$. Thus by choosing different β satisfying $0 < \beta \leq 1$ we subdivide Π_α into subregions $\Pi_{\alpha,\beta}$ such that $\Pi_{\alpha,\beta} \subset \Pi_{\alpha,\beta'}$ for $\beta \leq \beta'$. Also $\Pi_{\alpha,1} = \Pi_\alpha$. Further, it can be easily checked that, given any disc K in the half-plane Π_0, with centre on the real axis and which includes the point B in its interior, α and β satisfying $0 \leq \alpha < 1$, $0 < \beta \leq 1$ can be found such that $\Pi_{\alpha,\beta} = K$.

Now consider functions $f(z)$, given by (2.1.1), such that for all $z \in \Delta$, $zf'(z)/f(z)$ lies in some $\Pi_{\alpha,\beta}$ for some β . Since for every $\beta(0 < \beta \leq 1)$, $\Pi_{\alpha,\beta} \subset \Pi_\alpha$, it follows that each such function is starlike of order $\alpha(0 \leq \alpha < 1)$. We say that f is starlike of order $\alpha(0 \leq \alpha < 1)$ and type $\beta(0 < \beta \leq 1)$ if it maps the unit disc Δ into $\Pi_{\alpha,\beta}$. Thus we have the following definition:

Definition 2.1.1: Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc Δ. Then f is said to be starlike of order $\alpha (0 \leq \alpha < 1)$ and type $\beta (0 < \beta \leq 1)$ if

$$(2.1.7) \qquad \left| \left(z \frac{f'(z)}{f(z)} - 1\right) / \left\{2\beta\left(z\frac{f'(z)}{f(z)} - \alpha\right) - \left(z\frac{f'(z)}{f(z)} - 1\right)\right\}\right| < 1$$

holds for some $\alpha, \beta (0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ and for all $z \in \Delta$.

The class of starlike functions of order $\alpha (0 \leq \alpha < 1)$ and type $\beta (0 < \beta \leq 1)$ we shall denote by $S^*(\alpha, \beta)$.

It is easy to check that $S^*(\alpha, 1)$ is the class of starlike functions of order α ; $S^*(0,1)$ gives the whole class of starlike functions while $S^*(\alpha, 1/2)$ is the subclass of starlike functions studied by Wright [83] and McCarty [53]. The cases $(\alpha, \beta) = (0, 1/2)$; $(\alpha, \beta) = (0, (2\delta - 1)/2\delta)$ and $(\alpha, \beta) = ((1-\gamma)/(1+\gamma), (1+\gamma)/2)$ lead respectively to the classes introduced by Ram Singh [68,69] and Padmanabhan [61]. Also replacement of α by $1-\alpha$ and β by $1/2$ gives the class introduced by Eenigenburg [13].

Remark 2.1.1 : The introduction of the concept of 'type' for the class of starlike functions of order $\alpha (0 \leq \alpha < 1)$ gives a motivation to define the concept of 'type' for various existing subclasses of univalent functions. Thus we have the following.

Definition 2.1.2 : A function $p \in P$ is said to be of order $\alpha (0 \leq \alpha < 1)$ and type $\beta (0 < \beta \leq 1)$ in Δ if the inequality

(2.1.8) $$|(p(z)-1)/\{2\beta(p(z)-\alpha)-(p(z)-1)\}| < 1$$

holds for some α, $\beta(0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ and for all $z \in \Delta$.

In (2.1.8), replacing the function p by $zf'(z)/f(z)$, $1+zf''(z)/f'(z)$, $e^{i\alpha} f'(z)/\phi'(z)$, α-real, $\phi \in C$, $zf'(z)/\{f(z)-f(-z)\}$, $z(zf'(z))'/\{zf'(z)+zf'(-z)\}$, $e^{i\gamma}\, zf'(z)/f(z)$, $|\gamma|<\pi/2$ or $(1-\delta)zf'(z)/f(z) + \delta(1+zf''(z)/f'(z))$, $0 \leq \delta \leq 1$, we get respectively the class of starlike, convex, close-to-convex, starlike w.r.t. symmetric points [72], convex w.r.t. symmetric points [72], γ-spiral or δ-convex functions [54] of order $\alpha(0 \leq \alpha < 1)$ and type $\beta(0 < \beta \leq 1)$. However, in the present work, we shall confine ourselves to the cases $p(z)=zf'(z)/f(z)$ and $p(z) = 1+zf''(z)/f'(z)$ only and will not take up other classes. We shall denote the class of convex functions of order $\alpha(0 \leq \alpha < 1)$ and type $\beta(0 < \beta \leq 1)$ by $C(\alpha,\beta)$.

Remark 2.1.2: From the definitions of $S^*(\alpha,\beta)$ and $C(\alpha,\beta)$ it is obvious that $f \in C(\alpha,\beta)$ if, and only if, $zf' \in S^*(\alpha,\beta)$.

Remark 2.1.3 : The function $f(z)$, given by (2.1.1) and satisfying

(2.1.9) $$\left|\left(z\frac{f'(z)}{f(z)} - 1\right)\Big/\left(z\frac{f'(z)}{f(z)} + 1-2\alpha\right)\right| < \beta$$

for some α, $\beta(0 \leq \alpha < 1,\ 0 < \beta \leq 1)$, $z \in \Delta$ is easily seen to be of order $(1-\beta+2\alpha\beta)/(1+\beta)$ and type $(1+\beta)/2$. The class of functions satisfying (2.1.9) was introduced in [28].

As noticed above, our class $S^*(\alpha,\beta)$ includes the various subclasses of starlike functions. Hence, a study of its various properties will lead to a unified study of these subclasses. However, to include also the study of the subclasses $S^*_{\alpha,k}$, $\bar{S}_k(\delta)$, $S^*_k(\gamma)$, $\bar{S}^*_k(1-\alpha)$ and $\bar{S}^*_k(\alpha)$ of the class of starlike functions as defined in sec. 1.6, we shall, in the present chapter, study functions of $S^*(\alpha,\beta)$ whose power series begins $z+a_{k+1}z^{k+1}+a_{k+2}z^{k+2}+\dots$ We denote the class of all such functions by $S^*_k(\alpha,\beta)$. A representation formula, distortion theorems, coefficient estimates etc. are obtained for the functions in $S^*_k(\alpha,\beta)$. We also obtain a sufficient condition for a function to be in $S^*_k(\alpha,\beta)$. For different values of the parameters $\alpha,\beta(0 \leq \alpha < 1, 0 < \beta \leq 1)$, our results yield the corresponding results for their respective classes obtained by Schild [73, 74], MacGregor [42, 43], Boyd [8] , Ram Singh [68, 69], Padmanabhan [61] , Wright [83] , Eenigenburg [13] and McCarty [53] etc.

2.2 A representation formula for functions in $S^*_k(\alpha,\beta)$.

Let A denote the class of functions ϕ which are analytic in the unit disc Δ and $|\phi(z)| \leq 1$ for all $z \in \Delta$. We require the following lemma.

Lemma 2.2.1 : *Let* $H(z) = 1 + c_k z^k + c_{k+1} z^{k+1} + \dots$ *be analytic and satisfy the condition*

(2.2.1) $\quad |(H(z)-1)/\{2\beta(H(z)-\alpha)-(H(z)-1)\}| < 1, \quad (0 \leq \alpha < 1, 0 < \beta \leq 1)$

for all $z \in \Delta$. Then we have

$$(2.2.2) \qquad H(z) = \frac{1 + (2\alpha\beta-1)\, z^k \phi(z)}{1 + (2\beta-1)\, z^k \phi(z)}$$

where $\phi \in A$. Conversely any function H given by the formula (2.2.2) where $\phi \in A$ is analytic in Δ and satisfies (2.2.1) for all $z \in \Delta$.

Proof : Let $H(z) = 1 + c_k z^k + c_{k+1} z^{k+1} + \ldots$. Setting

$$(2.2.3) \qquad h(z) = \frac{1 - H(z)}{2\beta(H(z)-\alpha)-(H(z)-1)} .$$

We note that h is analytic and satisfies $|h(z)| < 1$ for all $z \in \Delta$ and also $h(z)$ has a zero of order k at $z = 0$. Thus by ([9] , pp. 138),

$$(2.2.4) \qquad h(z) = z^k \phi(z)$$

where $\phi \in A$. Hence from (2.2.3) and (2.2.4), we have

$$H(z) = \frac{1 + (2\alpha\beta-1)\, z^k \phi(z)}{1 + (2\beta-1) z^k\, \phi(z)} .$$

Also if $H(z)$ is given by (2.2.2), since

$$|z^k \phi(z)| \leq |z|^k < 1 \quad \text{in } \Delta \ ;$$

clearly H is analytic in Δ. The function

$$w = \frac{1 + (2\alpha\beta-1)\, z^k}{1 + (2\beta-1)\, z^k}$$

maps $|z| < 1$ onto the disc $|(1-w)/\{2\beta(w-\alpha)-(w-1)\}| < 1$ in w-plane and the converse part in the lemma follows from the above observation.

Theorem 2.2.1 : Let $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ be analytic in the unit disc Δ. Then $f \in S_k^*(\alpha,\beta)$ if and only if

$$f(z) = z \exp \left\{-2\beta(1-\alpha) \int_0^z \frac{t^{k-1}\phi(t)}{1 + (2\beta-1)\, t^k \phi(t)}\, dt\right\} \tag{2.2.5}$$

for some $\phi \in A$.

Proof : Let $f \in S_k^*(\alpha, \beta)$, it is easily seen that $zf'(z)/f(z)$ satisfies the hypothesis of the first part of Lemma 2.2.1. Therefore we can write

$$z\frac{f'(z)}{f(z)} = \frac{1 + (2\alpha\beta-1)\, z^k \phi(z)}{1 + (2\beta-1)\, z^k \phi(z)}$$

where $\phi \in A$, $z \in \Delta$. Thus, we have

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = -\frac{2\beta(1-\alpha)\, z^{k-1}\phi(z)}{1 + (2\beta-1)\, z^k \phi(z)} . \tag{2.2.6}$$

Integration gives (2.2.5) immediately. Conversely, if f has the representation (2.2.5) for some $\phi \in A$ then, it follows that

$$z\frac{f'(z)}{f(z)} = \frac{1 + (2\alpha\beta-1)\, z^k \phi(z)}{1 + (2\beta-1)\, z^k \phi(z)}$$

so that by converse part of Lemma 2.2.1, we have $f \in S_k^*(\alpha, \beta)$. Hence the Theorem.

For $k = 1$, we deduce the following :

Corollary 2.2.1a : Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc Δ. Then $f \in S^*(\alpha,\beta)$ if and only if

$$(2.2.7) \qquad f(z) = z \exp\left\{-2\beta(1-\alpha) \int_0^z \frac{\phi(t)}{1 + (2\beta-1)t\phi(t)}\, dt\right\}$$

for some $\phi \in A$.

Corollary 2.2.1b : Let $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ be analytic in the unit disc Δ. Then $f \in S^*_{\alpha,k}$ if and only if

$$(2.2.8) \qquad f(z) = z \exp\left\{-2(1-\alpha) \int_0^z \frac{t^{k-1}\phi(t)}{1 + t^k \phi(t)}\, dt\right\}$$

for some $\phi \in A$.

The above result is obtained by putting $\beta = 1$ in Theorem 2.2.1 and gives the representation formula for starlike functions of order $\alpha(0 \leq \alpha < 1)$.

Replacing α by $(1-\gamma)/(1+\gamma)$ and β by $(1+\gamma)/2$ in Theorem 2.2.1, we get the following representation formula for the functions of the class $S^*_k(\gamma)$.

Corollary 2.2.1c : Let $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ be analytic in the unit disc Δ. Then $f \in S^*_k(\gamma)$ if and only if

$$(2.2.9) \qquad f(z) = z \exp\left\{-2\gamma \int_0^z \frac{t^{k-1}\phi(t)}{1+\gamma t^k \phi(t)}\, dt\right\}$$

for some $\phi \in A$.

<u>Remark 2.2.1</u> : (a) Putting $\beta = 1/2$ or $(\alpha,\beta) = (0, 1/2)$ or $(\alpha,\beta) = (0, (2\delta-1)/2\delta)$ or replacing α by $1-\alpha$ and β by $1/2$ in Theorem 2.2.1, we deduce respectively the corresponding representation formulae for the functions in $\bar{S}_k^*(1-\alpha)$, $\bar{S}_k(1) \equiv \bar{S}_k^*$, $\bar{S}_k(\delta)$ and $\bar{S}_k^*(\alpha)$.

(b) Taking different values of the parameters $\alpha,\beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$ in Corollary 2.2.1a, we get corresponding representation formulae for the different subclasses of starlike functions obtained by Schild [74], MacGregor [43], Ram Singh [68, 69], Padmanabhan [61], Eenigenburg [13], McCarty [53] etc.

We now prove a corresponding result for the class $C_k(\alpha,\beta)$ of functions in $C(\alpha,\beta)$ whose power series begins $z + a_{k+1}z^{k+1} + \ldots$ as follows.

<u>Theorem 2.2.2</u> : <u>Let</u> $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ <u>be analytic in the unit disc</u> Δ. <u>Then</u> $f \in C_k(\alpha,\beta)$ <u>if and only if</u>

$$(2.2.10) \qquad f'(z) = \exp\left\{-2\beta(1-\alpha)\int_0^z \frac{t^{k-1}\phi(t)}{1+(2\beta-1)t^k\phi(t)}\,dt\right\}$$

<u>for some</u> $\phi \in A$.

<u>Proof</u> : Using the fact $f \in C_k(\alpha,\beta)$ if, and only if, $zf' \in S_k^*(\alpha,\beta)$ which is obvious by the definitions of $C_k(\alpha,\beta)$ and $S_k^*(\alpha,\beta)$, the theorem follows easily.

For $k = 1$, we deduce the following result:

Corollary 2.2.2 : Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc Δ. Then $f \in C(\alpha,\beta)$ if and only if

$$(2.2.11) \qquad f'(z) = \exp\{-2\beta(1-\alpha) \int_0^z \frac{\phi(t)}{1+(2\beta-1)\,t\phi(t)}\,dt\}$$

for some $\phi \in A$.

2.3 Distortion theorems.

Theorem 2.3.1 : Let $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ be analytic in the unit disc Δ and suppose that $f \in S_k^*(\alpha,\beta)$. Then we have, for $0 \leq \alpha < 1$, $\beta \neq 1/2$, $z \in \Delta$

$$(2.3.1) \qquad |f(z)| \leq \frac{|z|}{(1-(2\beta-1)|z|^k)^{2\beta(1-\alpha)/(2\beta-1)k}}$$

$$(2.3.2) \qquad |f(z)| \geq \frac{|z|}{(1-(1-2\beta)|z|^k)^{2\beta(1-\alpha)/(2\beta-1)k}}$$

whereas for $0 \leq \alpha < 1$, $\beta = 1/2$, $z \in \Delta$

$$(2.3.3) \qquad |f(z)| \leq |z| \exp\{\frac{(1-\alpha)}{k} |z|^k\}$$

$$(2.3.4) \qquad |f(z)| \geq |z| \exp\{-\frac{(1-\alpha)}{k} |z|^k\}.$$

All the above estimates are sharp.

Proof : Since $f \in S_k^*(\alpha,\beta)$, we observe that the condition (2.1.7), oupled with an application of Schwarz's Lemma [55] implies that for $\in \Delta$, $zf'(z)/f(z)$ assumes values lying in the disk K obtained by aking the line segment joining the points $(1+(2\alpha\beta-1)|z|^k)/$ $1+(2\beta-1)|z|^k)$ and $(1-(2\alpha\beta-1)|z|^k)/(1-(2\beta-1)|z|^k)$ as diameter.

Hence we have

$$(2.3.5)\qquad \frac{1+(2\alpha\beta-1)|z|^k}{1+(2\beta-1)\,|z|^k} \le \operatorname{Re}\ \{z\,\frac{f'(z)}{f(z)}\} \le \frac{1-(2\alpha\beta-1)|z|^k}{1-(2\beta-1)|z|^k}\ .$$

Let $|z| = r$, then (2.3.5) gives

$$\log\,(|\frac{f(z)}{z}|) = \operatorname{Re}\,(\log\,(\frac{f(z)}{z}))$$

$$= \operatorname{Re}\int_0^z \left[\frac{f'(s)}{f(s)} - \frac{1}{s}\right] ds$$

$$= \operatorname{Re}\int_0^z \frac{1}{s}\left[s\,\frac{f'(s)}{f(s)} - 1\right] ds$$

$$= \operatorname{Re}\int_0^{|z|} \frac{1}{t}\left[te^{i\theta}\,\frac{f'(te^{i\theta})}{f(te^{i\theta})} - 1\right] dt$$

$$= \int_0^{|z|} \frac{1}{t}\ \operatorname{Re}\,\{te^{i\theta}\,\frac{f'(te^{i\theta})}{f(te^{i\theta})} - 1\}\ dt$$

$$(2.3.6)\qquad \le \int_0^{|z|} \frac{2\beta(1-\alpha)t^{k-1}}{1-(2\beta-1)t^k}\ dt.$$

Now two cases arise . (i) If $0 \le \alpha < 1$, $\beta \ne 1/2$, then (2.3.6) gives

$$\log\,(|\frac{f(z)}{z}|) \le -\ \frac{2\beta(1-\alpha)}{(2\beta-1)k}\ \log\,(1-(2\beta-1)|z|^k).$$

Thus we have

$$|\frac{f(z)}{z}| \le \frac{1}{(1-(2\beta-1)|z|^k)^{2\beta(1-\alpha)/(2\beta-1)k}}$$

which gives (2.3.1).

(ii) If $0 \leq \alpha < 1, \beta = 1/2$, then (2.3.6) gives

$$\log\left(\left|\frac{f(z)}{z}\right|\right) \leq (1-\alpha) \int_0^{|z|} t^{k-1}\, dt$$

$$= (1-\alpha) \frac{|z|^k}{k}$$

This proves (2.3.3). To prove the remaining estimates, (2.3.5) gives

$$r \operatorname{Re} \left\{\frac{\partial}{\partial r} \left(\log \frac{f(z)}{z}\right)\right\} = \operatorname{Re} \left\{z \frac{f'(z)}{f(z)} - 1\right\}$$

$$\geq - \frac{2\beta(1-\alpha)|z|^k}{1+(2\beta-1)|z|^k} .$$

Thus we have

$$\log\left(\left|\frac{f(z)}{z}\right|\right) = \operatorname{Re}\left(\log\left(\frac{f(z)}{z}\right)\right)$$

(2.3.7)
$$\geq \int_0^r \frac{-2\beta(1-\alpha)t^{k-1}}{1+(2\beta-1)\, t^k}\, dt.$$

Again two cases arise . (i) If $0 \leq \alpha < 1$, $\beta \neq 1/2$, then (2.3.7) gives

$$\log\left(\left|\frac{f(z)}{z}\right|\right) \geq - \frac{2\beta(1-\alpha)}{(2\beta-1)k} \log\left(1+(2\beta-1)|z|^k\right) .$$

This proves (2.3.2).

(ii) If $0 \leq \alpha < 1, \beta = 1/2$, then (2.3.7) gives

$$\log\left(\left|\frac{f(z)}{z}\right|\right) \geq -(1-\alpha) \int_0^r t^{k-1}\, dt$$

$$= -(1-\alpha) \frac{|z|^k}{k}$$

which gives (2.3.4). Hence the Theorem.

Equality in (2.3.1) and (2.3.2) holds for the function

$$f(z) = \frac{z}{(1-(2\beta-1)\ z^{k})^{2\beta(1-\alpha)/(2\beta-1)k}}$$

whereas in (2.3.3) and (2.3.4) for the function

$$f(z) = z \exp \{\frac{1-\alpha}{k} z^{k}\} .$$

For $k = 1$, we deduce the following:

<u>Corollary 2.3.1a:</u> <u>Let</u> $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ <u>be analytic in the unit disc</u> Δ <u>and suppose that</u> $f \in S^*(\alpha, \beta)$. <u>Then we have, for</u> $0 \leq \alpha < 1,\ \beta \neq 1/2,\ z \in \Delta$

(2.3.8) $$|f(z)| \leq \frac{|z|}{(1-(2\beta-1)|z|)^{2\beta(1-\alpha)/(2\beta-1)}}$$

(2.3.9) $$|f(z)| \geq \frac{|z|}{(1+(2\beta-1)|z|)^{2\beta(1-\alpha)/(2\beta-1)}}$$

<u>whereas for</u> $0 \leq \alpha < 1,\ \beta = 1/2,\ z \in \Delta$

(2.3.10) $$|f(z)| \leq |z| \exp \{(1-\alpha)|z|\}.$$

(2.3.11) $$|f(z)| \geq |z| \exp \{-(1-\alpha)|z|\}.$$

<u>Equality in</u> (2.3.8) <u>and</u> (2.3.9) <u>holds for the function</u>

$$f(z) = \frac{z}{(1-(2\beta-1)z)^{2\beta(1-\alpha)/(2\beta-1)}}$$

<u>and in</u> (2.3.10) <u>and</u> (2.3.11) <u>for the function</u>

$$f(z) = ze^{(1-\alpha)z} .$$

Corollary 2.3.1b : Let $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ be analytic in the unit disc Δ and suppose that $f \in S^*_{\alpha,k}$. Then we have, for $z \in \Delta$

$$(2.3.12) \qquad |f(z)| \leq \frac{|z|}{(1-|z|^k)^{2(1-\alpha)/k}}$$

$$(2.3.13) \qquad |f(z)| \geq \frac{|z|}{(1+|z|^k)^{2(1-\alpha)/k}} .$$

The estimates are sharp.

The above result is obtained by putting $\beta = 1$ in Theorem 2.3.1 and gives the distortion theorems for starlike functions of order $\alpha(0 \leq \alpha < 1)$.

Replacing α by $(1-\gamma)/(1+\gamma)$ and β by $(1+\gamma)/2$ in Theorem 2.3.1, we get the following distortion theorems for the functions in $S^*_k(\gamma)$.

Corollary 2.3.1c : Let $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ be analytic in the unit disc Δ and suppose that $f \in S^*_k(\gamma)$. Then we have, for $z \in \Delta$

$$(2.3.14) \qquad |f(z)| \leq \frac{|z|}{(1-\gamma|z|^k)^{2/k}}$$

$$(2.3.15) \qquad |f(z)| \geq \frac{|z|}{(1+\gamma|z|^k)^{2/k}} .$$

The estimates are sharp.

Remark 2.3.1 : (a) Putting $\beta = 1/2$; $(\alpha,\beta) = (0, 1/2)$; $(\alpha,\beta) = (0,(2\delta-1)/2\delta)$ and replacement of α by $1-\alpha$ and β by $1/2$, in Theorem 2.3.1, we deduce respectively the corresponding distortion theorems for the functions in $\bar{S}_k^*(1-\alpha)$, $\bar{S}_k(1) \equiv \bar{S}_k^*$, $\bar{S}_k(\delta)$ and $\bar{S}_k^*(\alpha)$.

(b) Taking different values of the parameters α, β $(0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ in corollary 2.3.1a, we get corresponding distortion theorems for their respective classes obtained by Schild [74] , MacGregor [43] , Ram Singh [68,69] , Padmanabhan [61] , Eenigenburg [13] and McCarty [53] .

Theorem 2.3.1, together with the fact, $f \in C_k(\alpha,\beta)$ if, and only if, $zf' \in S_k^*(\alpha,\beta)$ gives the following corresponding result for the class $C_k(\alpha,\beta)$.

Theorem 2.3.2 : Let $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ be analytic in the unit disc Δ and suppose that $f \in C_k(\alpha,\beta)$. Then we have, for $0 \leq \alpha < 1$, $\beta \neq 1/2$, $z \in \Delta$

$$(2.3.16) \qquad |f'(z)| \leq \frac{1}{(1-(2\beta-1)|z|^k)^{2\beta(1-\alpha)/(2\beta-1)k}}$$

$$(2.3.17) \qquad |f'(z)| \geq \frac{1}{(1+(2\beta-1)|z|^k)^{2\beta(1-\alpha)/(2\beta-1)k}}$$

whereas for $0 \leq \alpha < 1$, $\beta = 1/2$, $z \in \Delta$

$$(2.3.18) \qquad |f'(z)| \leq \exp \left\{\frac{(1-\alpha)}{k} |z|^k\right\}$$

$$(2.3.19) \qquad |f'(z)| \geq \exp\left\{-\frac{(1-\alpha)}{k} |z|^k\right\} .$$

Equality in (2.3.16) and (2.3.17) holds for the function given by

$$(2.3.20) \qquad f'(z) = \frac{1}{(1-(2\beta-1)z^k)^{2\beta(1-\alpha)/(2\beta-1)k}}$$

whereas in (2.3.18) and (2.3.19) for the function given by

$$(2.3.21) \qquad f'(z) = \exp \left\{\frac{(1-\alpha)}{k} z^k\right\} .$$

For k = 1, we deduce the following result :

Corollary 2.3.2 : Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc Δ and suppose that $f \in C(\alpha,\beta)$. Then we have, for $0 \leq \alpha < 1$, $\beta \neq 1/2$, $z \in \Delta$

$$(2.3.22) \qquad |f'(z)| \leq \frac{1}{(1-(2\beta-1)|z|)^{2\beta(1-\alpha)/(2\beta-1)}}$$

$$(2.3.23) \qquad |f'(z)| \geq \frac{1}{(1+(2\beta-1)|z|)^{2\beta(1-\alpha)/(2\beta-1)}}$$

whereas for $0 \leq \alpha < 1$, $\beta = 1/2$, $z \in \Delta$

$$(2.3.24) \qquad |f'(z)| \leq \exp \{(1-\alpha)|z|\}$$

$$(2.3.25) \qquad |f'(z)| \geq \exp \{(\alpha-1)|z|\} .$$

Equality in the above estimates holds for the functions given by (2.3.20) and (2.3.21) with k = 1.

2.4 <u>A sufficient condition for a function to be in</u> $S_k^*(\alpha,\beta)$ <u>and</u> $C_k(\alpha,\beta)$.

<u>Theorem 2.4.1</u> : <u>Let</u> $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ <u>be analytic in the unit disc</u> Δ. <u>If for some</u> $\alpha,\beta (0 \leq \alpha < 1,\ 0 < \beta \leq 1/2)$,

$$\sum_{n=k+1}^{\infty} [(1-\beta)n - \alpha\beta]|a_n| \leq \beta(1-\alpha), \tag{2.4.1}$$

<u>then</u> $f(z)$ <u>belongs to</u> $S_k^*(\alpha,\beta)$.

<u>Proof</u> : We employ the same technique as used by Clunie and Keogh [12]. Thus suppose that (2.4.1) holds and that

$$f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n ;$$

then for $z \in \Delta$,

$$|zf'(z) - f(z)| - |2\beta(zf'(z) - \alpha f(z)) - (zf'(z) - f(z))|$$

$$= |\sum_{n=k+1}^{\infty} (n-1)a_n z^n| - |2\beta\{(1-\alpha)z + \sum_{n=k+1}^{\infty} (n-\alpha)a_n z^n\} - \sum_{n=k+1}^{\infty} (n-1)\, a_n z^n|$$

$$= |\sum_{n=k+1}^{\infty} (n-1)a_n z^n| - |2\beta(1-\alpha)z + \sum_{n=k+1}^{\infty} (1-2\alpha\beta)\, a_n z^n + \sum_{n=k+1}^{\infty} (2\beta-1)n\, a_n z^n|$$

$$\leq \sum_{n=k+1}^{\infty} (n-1)|a_n|r^n - \{|2\beta(1-\alpha)z + \sum_{n=k+1}^{\infty} (1-2\alpha\beta)a_n z^n| - \sum_{n=k+1}^{\infty} (1-2\beta)n\, |a_n|r^n\}$$

$$\leq \sum_{n=k+1}^{\infty} (n-1)|a_n|r^n - \{2\beta(1-\alpha)r - \sum_{n=k+1}^{\infty} (1-2\alpha\beta)|a_n|r^n - \sum_{n=k+1}^{\infty} (1-2\beta)n|a_n|r^n\}$$

$$< [\sum_{n=k+1}^{\infty} (n-1)|a_n| - 2\beta(1-\alpha) + \sum_{n=k+1}^{\infty} (1-2\alpha\beta+n-2\beta n)|a_n|]\ r$$

$$= [\sum_{n=k+1}^{\infty} [(1-\beta)n - \alpha\beta]\ |a_n| - \beta(1-\alpha)]\ 2r$$

≤ 0, by (2.4.1).

Hence it follows that

$$\left|(z\frac{f'(z)}{f(z)} - 1) / \{2\beta\ (z\frac{f'(z)}{f(z)} - \alpha) - (z\frac{f'(z)}{f(z)} - 1)\}\right| < 1,$$

so that $f \in S_k^*(\alpha,\beta)$.

For $k = 1$, we deduce the following result :

<u>Corollary 2.4.1</u> : <u>Let</u> $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ <u>be analytic in the unit disc</u> Δ. <u>If for some</u> α,β $(0 \leq \alpha < 1,\ 0 < \beta \leq 1/2)$,

$$(2.4.2) \qquad \sum_{n=2}^{\infty} [(1-\beta)n - \alpha\beta]\ |a_n| \leq \beta(1-\alpha),$$

<u>then</u> $f(z)$ <u>belongs to</u> $S^*(\alpha,\beta)$.

<u>Remark 2.4.1</u> : (a) Taking $\beta = 1/2$; $(\alpha,\beta) = (0,\ 1/2)$; $(\alpha,\beta) = (0,\ (2\delta-1)/2\delta)$ and replacing α by $1-\alpha$ and β by $1/2$, in Theorem 2.4.1, we get respectively a sufficient condition for functions in $\bar{S}_k^*(1-\alpha)$, $\bar{S}_k(1) \equiv \bar{S}_k^*$, $\bar{S}_k(\delta)$ and $\bar{S}_k^*(\alpha)$.

(b) For different values of the parameters $\alpha, \beta (0 \le \alpha < 1, 0 < \beta \le 1/2)$ in corollary 2.4.1, we get corresponding sufficient conditions for their respective classes **studied** by Ram Singh [68,69], Eenigenburg [13] and McCarty [53].

We now give a corresponding sufficient condition for functions in $C_k(\alpha, \beta)$.

Theorem 2.4.2 : Let $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ be analytic in the unit disc Δ. If for some $\alpha, \beta (0 \le \alpha < 1, 0 < \beta \le 1/2)$,

$$\sum_{n=k+1}^{\infty} n\,[(1-\beta)n - \alpha\beta]\,|a_n| \le (1-\alpha)\beta, \tag{2.4.3}$$

then $f(z)$ belongs to $C_k(\alpha, \beta)$.

Proof : The proof follows immediately by the fact, $f \in C_k(\alpha,\beta)$ if, and only if, $zf' \in S_k^*(\alpha,\beta)$, since we may replace a_n by $n\,a_n$ in Theorem 2.4.1.

For $k = 1$, we deduce the following result:

Corollary 2.4.2: Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc Δ. If for some $\alpha, \beta (0 \le \alpha < 1, 0 < \beta \le 1/2)$,

$$\sum_{n=2}^{\infty} n\,[(1-\beta)n - \alpha\beta]\,|a_n| \le (1-\alpha)\beta, \tag{2.4.4}$$

then $f(z)$ belongs to $C(\alpha, \beta)$.

2.5 Some coefficient estimates:

Theorem 2.5.1: Let $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ be in $S_k^*(\alpha, \beta)$.

(a) *If* $\beta(1-\alpha) > k(1-\beta)$, *let* $M = [\frac{\beta(1-\alpha)}{k(1-\beta)}]$. *Then*

$$(2.5.1) \qquad |a_n| \leq \frac{k}{(n-1)(m-1)!} \prod_{\mu=0}^{m-1} ((2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k})$$

for $mk+1 \leq n \leq (m+1)k$, $m = 1,2,\ldots, M+1$ *and*

$$(2.5.2) \qquad |a_n| \leq \frac{k}{(n-1)(M+1)!} \prod_{\mu=0}^{M+1} ((2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k}), \; n > (M+2)k.$$

(b) *If* $\beta(1-\alpha) \leq k(1-\beta)$, *then*

$$(2.5.3) \qquad |a_n| \leq \frac{2\beta(1-\alpha)}{n-1} \quad \text{for} \quad n \geq k+1.$$

The estimates in (2.5.1) *are sharp for* $n = mk+1$, $m = 1,2,\ldots$ *while the estimates in* (2.5.3) *are sharp for all* n.

Proof: We employ the technique used by MacGregor [42] . Thus, let $f \in S_k^*(\alpha,\beta)$, then we have

$$(2.5.4) \qquad h(z) = \frac{zf'(z) - f(z)}{2\beta(zf'(z)-\alpha f(z))-(zf'(z)-f(z))}$$

where h is regular in Δ and satisfies $|h(z)| < 1$ in Δ . Also the power series for $h(z)$ begins with $c_k z^k + c_{k+1} z^{k+1} + \ldots$. Equating coefficients of the same powers on both sides of the equation

$$zf'(z) - f(z) = h(z)\{2\beta(zf'(z) - \alpha f(z)) - (zf'(z)-f(z))\}$$

or

$$(2.5.5)\qquad \sum_{n=k+1}^{\infty}(n-1)\,a_n z^n = \{c_k z^k + c_{k+1}z^{k+1}+\dots\}\,\{2\beta(1-\alpha)\,z + \sum_{n=k+1}^{\infty}((2\beta-1)n+1-2\alpha\beta)a_n z^n\}$$

we obtain

$$(2.5.6)\qquad (n-1)\,a_n = 2\beta(1-\alpha)c_{n-1} \quad \text{for } n = k+1,\ k+2,\dots,2k.$$

Since $|h(z)| < 1$, it follows that $\sum_{n=k}^{\infty}|c_n|^2 \leq 1$ and so

$$(2.5.7)\qquad \sum_{n=k}^{2k-1}|c_n|^2 \leq 1.$$

From (2.5.6) and (2.5.7), we find that

$$(2.5.8)\qquad \sum_{n=k+1}^{2k}(n-1)^2|a_n|^2 \leq 4\beta^2(1-\alpha)^2.$$

(2.5.5) can be rewritten in the form

$$(2.5.9)\qquad \sum_{n=k+1}^{p}(n-1)a_n z^n + \sum_{n=p+1}^{\infty} d_n z^n = h(z)\{2\beta(1-\alpha)\,z + \sum_{n=k+1}^{p-k}((2\beta-1)n+1-2\alpha\beta)\,a_n z^n\}.$$

Since (2.5.9) has the form $F(z) = h(z)\,G(z)$, where $|h(z)| < 1$, it follows that

$$(2.5.10)\qquad \frac{1}{2\pi}\int_0^{2\pi}|F(re^{i\phi})|^2 d\phi \leq \frac{1}{2\pi}\int_0^{2\pi}|G(re^{i\phi})|^2\,d\phi$$

for each $r(0 < r < 1)$. Expressing (2.5.10) in terms of the coefficients in (2.5.9), we get

$$(2.5.11)\qquad \sum_{n=k+1}^{p}(n-1)^2|a_n|^2 r^{2n} + \sum_{n=p+1}^{\infty}|d_n|^2 r^{2n} \leq \{4\beta^2(1-\alpha)^2 r^2 + \sum_{n=k+1}^{p-k}((2\beta-1)n+1-2\alpha\beta)^2|a_n|^2 r^{2n}\}.$$

In particular, (2.5.11) implies

$$(2.5.12)\qquad \sum_{n=k+1}^{p} (n-1)^2 |a_n|^2 \, r^{2n} \leq 4\beta^2(1-\alpha)^2 r^2 + \sum_{n=k+1}^{p-k} ((2\beta-1)n + 1-2\alpha\beta)^2 |a_n|^2 \, r^{2n}.$$

Letting $r \to 1$ in (2.5.12), we conclude that

$$(2.5.13)\qquad \sum_{n=k+1}^{p} (n-1)^2 |a_n|^2 \leq 4\beta^2(1-\alpha)^2 + \sum_{n=k+1}^{p-k} ((2\beta-1)n+1-2\alpha\beta)^2 |a_n|^2.$$

This inequality is equivalent to

$$(2.5.14)\qquad \sum_{n=p-k+1}^{p} (n-1)^2 |a_n|^2 \leq 4\beta^2(1-\alpha)^2 + \sum_{n=k+1}^{p-k} \{((2\beta-1)n+1-2\alpha\beta)^2 - (n-1)^2\} |a_n|^2.$$

Now two cases arise. (a) if $\beta(1-\alpha) > k(1-\beta)$, then by an inductive argument we will establish the inequalities

$$(2.5.15;a)\qquad \sum_{n=mk+1}^{(m+1)k} (n-1)^2 |a_n|^2 \leq \left\{ \frac{k}{(m-1)!} \prod_{\mu=0}^{m-1} \left((2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k}\right) \right\}^2$$

$$(2.5.15;b)\qquad \sum_{n=mk+1}^{(m+1)k} \{((2\beta-1)\, n + 1-2\alpha\beta)^2 - (n-1)^2\} |a_n|^2$$

$$\leq \left\{ \frac{1}{m!} \prod_{\mu=0}^{m-1} \left((2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k}\right) \right\}^2 \{((2\beta-1)mk + 2\beta(1-\alpha))^2 - m^2k^2\}$$

for $m = 1, 2, \ldots, M+1$; $M = \left[\dfrac{\beta(1-\alpha)}{k(1-\beta)}\right]$ where $[p]$ denotes the greatest integer not greater than p.

For $m = 1$, (2.5.15;a) gives

$$\sum_{n=k+1}^{2k} (n-1)^2 |a_n|^2 \leq 4\beta^2(1-\alpha)^2$$

which is same as (2.5.8). Thus (2.5.15;a) is valid for $m = 1$. We can prove (2.5.15;b) for $m = 1$ by using (2.5.8) as follows.

$$\sum_{n=k+1}^{2k} \{((2\beta-1)\,n + 1-2\alpha\beta)^2 - (n-1)^2\}\, |a_n|^2$$

$$\leq \frac{\{(2\beta-1)k+2\beta(1-\alpha)\}^2 - k^2}{k^2} \sum_{n=k+1}^{2k} (n-1)^2 |a_n|^2$$

$$\leq \frac{4\beta^2(1-\alpha)^2}{k^2} \{((2\beta-1)\,k + 2\beta(1-\alpha))^2 - k^2\}\,.$$

Now suppose that (2.5.15;a) and (2.5.15;b) hold for $m = 1,2,\ldots,q-1$. Using (2.5.14) with $p = (q+1)k$ and the inductive hypothesis concerning (2.5.15;a), we obtain the inequalities

$$\sum_{n=qk+1}^{(q+1)k} (n-1)^2 |a_n|^2 \leq 4\beta^2(1-\alpha)^2 + \sum_{n=k+1}^{qk} \{((2\beta-1)n+1-2\alpha\beta)^2-(n-1)^2\}|a_n|^2$$

$$= 4\beta^2(1-\alpha)^2 + \sum_{m=1}^{q-1} \sum_{n=mk+1}^{(m+1)k} \{((2\beta-1)n+1-2\alpha\beta)^2-(n-1)^2\}|a_n|^2$$

$$\leq 4\beta^2(1-\alpha)^2 + \sum_{m=1}^{q-1} \{\frac{1}{m!} \prod_{\mu=0}^{m-1} ((2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k})\}^2 \times$$

$$\{((2\beta-1)mk+2\beta(1-\alpha))^2-m^2k^2\}$$

$$= \{\frac{k}{(q-1)!} \prod_{\mu=0}^{q-1} ((2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k})\}^2\,.$$

The last equality can be easily obtained by an inductive argument on q.

This last sequence of inequalities implies (2.5.15;a) where $m = q$.

Continuing our argument, we use (2.5.15;a) with $m = q$ to deduce (2.5.15;b) for $m = q$ as follows.

$$\sum_{n=qk+1}^{(q+1)k} \{((2\beta-1)n + 1 - 2\alpha\beta)^2 - (n-1)^2\}|a_n|^2$$

$$\leq \frac{((2\beta-1)qk + 2\beta(1-\alpha))^2 - q^2k^k}{q^2k^2} \sum_{n=qk+1}^{(q+1)k} (n-1)^2 \ |a_n|^2$$

$$\leq \frac{((2\beta-1)qk + 2\beta(1-\alpha))^2 - q^2k^2}{q^2k^2} \{\frac{k}{(q-1)!} \prod_{\mu=0}^{q-1} ((2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k})\}^2$$

$$= \{\frac{1}{q!} \prod_{\mu=0}^{q-1} ((2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k})\}^2\{((2\beta-1)qk+2\beta(1-\alpha))^2-q^2k^2\} .$$

This completes the proof of (2.5.15;a) and (2.5.15;b). Now (2.5.1) follows from (2.5.15;a).

To prove (2.5.2), suppose $n > (M+2)k$. Putting $p = (q+1)k$ in (2.5.14), we have

$$\sum_{n=qk+1}^{(q+1)k} (n-1)^2 |a_n|^2 \leq 4\beta^2(1-\alpha)^2 + \sum_{n=k+1}^{qk} \{((2\beta-1)n+1-2\alpha\beta)^2-(n-1)^2\} |a_n|^2 .$$

Hence, for $n > (M+2)k$, we have

$$(n-1)^2 |a_n|^2 \leq 4\beta^2(1-\alpha)^2 + \sum_{n=k+1}^{qk} \{((2\beta-1)n+1-2\alpha\beta)^2-(n-1)^2\}|a_n|^2$$

$$= 4\beta^2(1-\alpha)^2 + \sum_{n=k+1}^{(M+2)k} \{((2\beta-1)n+1-2\alpha\beta)^2-(n-1)^2\}|a_n|^2$$

$$+ \sum_{n=(M+2)k+1}^{qk} \{((2\beta-1)n+1-2\alpha\beta)^2-(n-1)^2\} |a_n|^2$$

$$= 4\beta^2(1-\alpha)^2 + \sum_{m=1}^{M+1} \sum_{n=mk+1}^{(m+1)k} \{((2\beta-1)n+1-2\alpha\beta)^2-(n-1)^2\} |a_n|^2$$

$$+ \sum_{m=M+2}^{q-1} \sum_{n=mk+1}^{(m+1)k} \{((2\beta-1)n+1-2\alpha\beta)^2-(n-1)^2\} |a_n|^2$$

$$(2.5.16) \leq 4\beta^2(1-\alpha)^2 + \sum_{m=1}^{M+1} \sum_{n=mk+1}^{(m+1)k} \{((2\beta-1)n+1-2\alpha\beta)^2-(n-1)^2\} |a_n|^2 .$$

Using (2.5.15;b) in (2.5.16), we obtain

$$(n-1)^2|a_n|^2 \leq \{\frac{k}{(M+1)!} \prod_{\mu=0}^{M+1} ((2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k})\}^2$$

i.e. $$|a_n| \leq \frac{k}{(n-1)\ (M+1)!} \prod_{\mu=0}^{M+1} ((2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k}),\ n > (M+2)\ k.$$

This proves (2.5.2).

(b) If $\beta(1-\alpha) \leq k(1-\beta)$, then (2.5.13) gives

$$\sum_{n=k+1}^{p} (n-1)^2\ |a_n|^2 \leq 4\beta^2(1-\alpha)^2$$

or $$(n-1)^2\ |a_n|^2 \leq 4\beta^2(1-\alpha)^2 \quad \text{if} \quad n \geq k+1,$$

i.e., $$|a_n| \leq \frac{2\beta(1-\alpha)}{(n-1)} \qquad \text{if} \quad n \geq k+1.$$

which gives (2.5.3).

The function f given by

$$z\frac{f'(z)}{f(z)} = \frac{1-(2\alpha\beta-1)\,z^k}{1-(2\beta-1)\,z^k} \quad \text{where} \quad \beta(1-\alpha) > k(1-\beta)$$

shows that the estimates in (2.5.1) are sharp for $n = mk+1$, $m = 1,2,3,\ldots$ while the estimates in (2.5.3) are sharp for the function

$$f(z) = z\exp\{[2\beta(1-\alpha)/(n-1)]\,z^{n-1}\},$$

where $\beta(1-\alpha) \leq k(1-\beta)$ and $n \geq k+1$.

For $k = 1$, we deduce the following result:

<u>Corollary 2.5.1a</u>: <u>Let</u> $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ <u>be in</u> $S^*(\alpha,\beta)$.

(a) <u>If</u> $\beta(1-\alpha) > 1-\beta$, let $M = \left[\frac{\beta(1-\alpha)}{1-\beta}\right]$. <u>Then</u>

(2.5.17) $$|a_n| \leq \frac{\prod_{k=2}^{n}((2\beta-1)k + 2(1-\beta-\alpha\beta))}{(n-1)!}$$

<u>for</u> $n = 2,3,\ldots,N \equiv M+2$ <u>and</u>

(2.5.18) $$|a_n| \leq \frac{1}{n-1}\,\frac{\prod_{k=2}^{N+1}((2\beta-1)k + 2(1-\beta-\alpha\beta))}{(N-1)!}, \quad n > N$$

(b) <u>If</u> $\beta(1-\alpha) \leq 1-\beta$, <u>then</u>

(2.5.19) $$|a_n| \leq \frac{2\beta(1-\alpha)}{n-1} \quad \text{for } n \geq 2.$$

<u>The estimates in</u> (2.5.17) <u>are sharp for the function given by</u>

$$z\frac{f'(z)}{f(z)} = \frac{1-(2\alpha\beta-1)z}{1-(2\beta-1)z} \quad \underline{\text{where}} \quad \beta(1-\alpha) > (1-\beta)$$

<u>while the estimates in</u> (2.5.19) <u>are sharp for the function</u>

$$f(z) = z\exp\{[2\beta(1-\alpha)/(n-1)]\ z^{n-1}\}, \text{where}\ \ \beta(1-\alpha) \leq 1-\beta \text{ and } n \geq 2.$$

Putting $\beta = 1$ in Theorem 2.5.1, we get the following result due to Boyd [8] .

<u>Corollary 2.5.1b</u> : <u>If</u> $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ <u>is in</u> $S^*_{\alpha,k}$.

<u>Then</u>

(2.5.20) $$|a_n| \leq \frac{k}{(n-1)(m-1)!}\prod_{\mu=0}^{m-1}\left(\mu+\frac{2(1-\alpha)}{k}\right)$$

<u>where</u> $mk+1 \leq n \leq (m+1)k$, $m = 1,2,\ldots\ldots$ <u>The result is sharp for</u> $n = mk+1$, $m = 1,2,\ldots\ldots$ <u>for the function</u>

$$f(z) = \frac{z}{(1-z^k)^{2(1-\alpha)/k}}\ .$$

The following result due to MacGregor [42] can be obtained by taking $(\alpha,\beta) = (0,1)$ in Theorem 2.5.1 or $\alpha = 0$ in corollary 2.5.1b.

<u>Corollary 2.5.1c</u>: <u>If</u> $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ <u>is in</u> S^*_k . <u>Then</u>

(2.5.21) $$|a_n| \leq \frac{k}{(n-1)(m-1)!}\prod_{\mu=0}^{m-1}(\mu + 2/k)$$

<u>where</u> $mk+1 \leq n \leq (m+1)k$, $m = 1,2,\ldots\ldots$. <u>The estimates are sharp for</u> $n = mk+1$ $(m = 1,2,\ldots\ldots)$, <u>for</u>

$$f(z) = \frac{z}{(1-z^k)^{2/k}} .$$

Remark 2.5.1:(a) Putting $\beta = 1/2$; $(\alpha,\beta) = (0,1/2)$; $(\alpha,\beta) = ((1-\gamma)/(1+\gamma), (1+\gamma)/2)$ and replacing α by $1-\alpha$ and β by $1/2$ in Theorem 2.5.1, we get respectively the corresponding coefficient estimates for the functions of the classes $\bar{S}_k^*(1-\alpha)$, $\bar{S}_k^*$, $S_k^*(\gamma)$ and $\bar{S}_k^*(\alpha)$.

(b) For different values of the parameters $\alpha, \beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$ in corollary 2.5.1a, we get corresponding coefficient estimates for their respective classes obtained by Schild [74], Ram Singh [68, 69], Eenigenburg [13], McCarty [53] etc.

We now prove a corresponding result for functions of the class $C_k(\alpha,\beta)$ as follows.

Theorem 2.5.2 : Let $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ be in $C_k(\alpha,\beta)$.

(a) If $\beta(1-\alpha) > k(1-\beta)$, let $M = \left[\frac{\beta(1-\alpha)}{k(1-\beta)}\right]$. Then

$$(2.5.22) \qquad |a_n| \leq \frac{k}{n(n-1)(m-1)!} \prod_{\mu=0}^{m-1} \left((2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k}\right)$$

for $mk+1 \leq n \leq (m+1)k$, $m = 1,2,\ldots, M+1$ and

$$(2.5.23) \qquad |a_n| \leq \frac{k}{n(n-1)(M+1)!} \prod_{\mu=0}^{M+1} \left((2\beta-1)\mu + \frac{2\beta(1-\alpha)}{k}\right), \; n > (M+2)k.$$

(b) If $\beta(1-\alpha) \leq k(1-\beta)$, then

$$(2.5.24) \qquad |a_n| \leq \frac{2\beta(1-\alpha)}{n(n-1)} \quad \text{for} \quad n \geq k+1.$$

The result (2.5.22) is sharp for $n = mk+1$, $m=1,2,\ldots$ for the function given by

$$1 + z\frac{f''(z)}{f'(z)} = \frac{1-(2\alpha\beta-1)z^k}{1-(2\beta-1)z^k} \quad \text{where} \quad \beta(1-\alpha) > k(1-\beta)$$

while the estimates in (2.5.24) are sharp for the function

$$f(z) = z \exp\left\{[2\beta(1-\alpha)/n(n-1)]z^{n-1}\right\}, \text{ where}$$

$$\beta(1-\alpha) \leq k(1-\beta), \; n \geq k+1.$$

For $k = 1$, we deduce the following result:

Corollary 2.5.2: Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in $C(\alpha,\beta)$.

(a) If $\beta(1-\alpha) > (1-\beta)$, let $M = \left[\frac{\beta(1-\alpha)}{1-\beta}\right]$. Then

$$(2.5.25) \qquad |a_n| \leq \frac{\prod_{k=2}^{n}((2\beta-1)k+2(1-\beta-\alpha\beta))}{n!}$$

for $n = 2,3,\ldots,N \equiv M+2$ and

$$(2.5.26) \qquad |a_n| \leq \frac{1}{n(n-1)} \frac{\prod_{k=2}^{N+1}((2\beta-1)k+2(1-\beta-\alpha\beta))}{(N-1)!}, \; n > N.$$

(b) If $\beta(1-\alpha) \leq (1-\beta)$, then

$$(2.5.27) \qquad |a_n| \leq \frac{2\beta(1-\alpha)}{n(n-1)}, \; n \geq 2.$$

The estimates in (2.5.25) are sharp for the function given by

$$1 + z\frac{f''(z)}{f'(z)} = \frac{1-(2\alpha\beta-1)z}{1-(2\beta-1)z} \quad \text{where} \quad \beta(1-\alpha) > (1-\beta)$$

while the estimates in (2.5.27) are sharp for the function

$$f(z) = z\exp\left\{[2\beta(1-\alpha)/n(n-1)]\ z^{n-1}\right\} \text{ where } \quad \beta(1-\alpha) \leq 1-\beta$$

$$\text{and} \quad n \geq 2.$$

CHAPTER III

STARLIKE FUNCTIONS OF ORDER α AND TYPE β - II

3.1 In this chapter, we first determine the radii of convexity for starlike functions of order $\alpha(0 \leq \alpha < 1)$ and type $\beta(0 < \beta \leq 1)$ as defined in Chapter II. This determination gives a unified approach to obtain the radii of convexity for the different subclasses of starlike functions that have been obtained earlier by Schild [74], Zmorovič [84], Singh and Goel [78], MacGregor [43], Ram Singh [68,69], Padmanabhan [61], Eenigenburg [13], McCarty [53] etc. We then determine the radii of convexity for the class $S_k^*(\alpha,\beta)$ of analytic functions whose power series begins $f(z) = z + a_{k+1}z^{k+1}+a_{k+2}z^{k+2}+\ldots$ and which are starlike of order α and type β. These results in particular, lead to some new results regarding radii of convexity for the classes introduced by Robertson [71], MacGregor [43], Ram Singh [68, 69], Padmanabhan [61], Wright [83] and Eenigenburg [13]. Finally, we obtain the radii of starlikeness for the class $V^*(\alpha,\beta)$ of functions of the form

$$(3.1.1) \qquad F(z) = 1/2\ (zf(z))'$$

where $f \in S^*(\alpha,\beta)$. The results thus obtained, besides yielding a few new results, also include the results obtained by Livingston [39], Libera and Livingston [37], Singh and Goel [78], Al-Amiri [2] and others.

3.2 Let $\mathcal{B}$ denote the class of functions ω analytic in Δ which satisfy (i) $\omega(0) = 0$ and (ii) $|\omega(z)| < 1$ for $z \in \Delta$. We first prove a few lemmas.

Lemma 3.2.1 [78]: If $\omega \in \mathcal{B}$, then for all $z \in \Delta$

$$(3.2.1) \qquad |z\omega'(z) - \omega(z)| \leq \frac{|z|^2 - |\omega(z)|^2}{1-|z|^2}.$$

Lemma 3.2.2: Let $\omega \in \mathcal{B}$. Then we have

$$(3.2.2) \qquad \mathrm{Re}\left\{\frac{z^k\omega'(z) + (k-1)z^{k-1}\omega(z)}{(1+sz^{k-1}\omega(z))(1+tz^{k-1}\omega(z))}\right\} \leq -\frac{k}{(s-t)^2}\mathrm{Re}\left\{sp(z) + \frac{t}{p(z)} - s - t\right\} + \frac{r^{2k}|sp(z)-t|^2 - |1-p(z)|^2}{(s-t)^2 r^{k-1}(1-r^2)|p(z)|}$$

$$(3.2.3) \qquad \mathrm{Re}\left\{\frac{z^k\omega'(z) + (k-1)z^{k-1}\omega(z)}{(1+sz^{k-1}\omega(z))(1+tz^{k-1}\omega(z))}\right\} \geq -\frac{k}{(s-t)^2}\mathrm{Re}\left\{sp(z) + \frac{t}{p(z)} - s - t\right\} - \frac{r^{2k}|sp(z)-t|^2 - |1-p(z)|^2}{(s-t)^2 r^{k-1}(1-r^2)|p(z)|}$$

where $p(z) = (1+tz^{k-1}\omega(z))/(1+sz^{k-1}\omega(z))$, $|z| = r$, $-1 \leq t < s \leq 1$ and $k \geq 1$ is a fixed positive integer.

Proof: Since $p(z) = (1+tz^{k-1}\omega(z))/(1+sz^{k-1}\omega(z))$, we have

$$(3.2.4) \qquad \omega(z) = \frac{1-p(z)}{z^{k-1}(sp(z)-t)}.$$

Simple calculations give

$$(3.2.5) \qquad \frac{1}{(1+sz^{k-1}\omega(z))(1+tz^{k-1}\omega(z))} = \frac{(sp(z)-t)^2}{(s-t)^2 p(z)}$$

and

$$(3.2.6) \qquad \frac{kz^{k-1}\omega(z)}{(1+sz^{k-1}\omega(z))(1+tz^{k-1}\omega(z))} = -\frac{k}{(s-t)^2}\{sp(z)+\frac{t}{p(z)}-s-t\} .$$

Therefore Lemma 3.2.1, equations (3.2.4) and (3.2.5) give

$$\left|\frac{z^k\omega'(z)+(k-1)z^{k-1}\omega(z)}{(1+sz^{k-1}\omega(z))(1+tz^{k-1}\omega(z))} - \frac{kz^{k-1}\omega(z)}{(1+sz^{k-1}\omega(z))(1+tz^{k-1}\omega(z))}\right|$$

$$\leq \frac{r^{2k}|sp(z)-t|^2-|1-p(z)|^2}{(s-t)^2\, r^{k-1}(1-r^2)|p(z)|}$$

or

$$(3.2.7) \qquad \operatorname{Re}\left\{\frac{z^k\omega'(z)+(k-1)z^{k-1}\omega(z)}{(1+sz^{k-1}\omega(z))(1+tz^{k-1}\omega(z))}\right\} \leq \operatorname{Re}\left\{\frac{kz^{k-1}\omega(z)}{(1+sz^{k-1}\omega(z))(1+tz^{k-1}\omega(z))}\right\}$$

$$+\frac{r^{2k}|sp(z)-t|^2-|1-p(z)|^2}{(s-t)^2\, r^{k-1}(1-r^2)|p(z)|} .$$

Hence (3.2.2) follows immediately from (3.2.6) and (3.2.7).

(3.2.3) can be proved in a similar manner.

<u>Remark 3.2.1</u>: <u>The transformation</u> $p(z) = (1+tz^{k-1}\omega(z))/(1+sz^{k-1}\omega(z))$ <u>maps the circle</u> $|\omega(z)| \leq r$ <u>onto the circle</u>

$$\left|p(z) - \frac{1-st\, r^{2k}}{1-s^2r^{2k}}\right| \leq \frac{(s-t)\, r^k}{1-s^2r^{2k}} .$$

Lemma 3.2.3: Let $p(z) = (1+tz^{k-1}\omega(z))/(1+sz^{k-1}\omega(z))$ where $\omega \in \mathcal{B}$, $A = (1-str^{2k})/(1-s^2r^{2k})$, $D = (s-t)\, r^k/(1-s^2r^{2k})$, then for $|z| = r$, $0 < r < 1$, we have

$$\mathrm{Re}\,\{qp(z) + \frac{tk}{p(z)}\} - \frac{r^{2k}|sp(z)-t|^2-|1-p(z)|^2}{r^{k-1}(1-r^2)|p(z)|}$$

$$(3.2.8)\quad \geq \begin{cases} \dfrac{2}{r^{k-1}(1-r^2)}\Big[\sqrt{(qr^{k-1}(1-r^2)+1-s^2r^{2k})(1+tkr^{k-1}(1-r^2)-t^2r^{2k})} \\ \qquad\qquad -(1-str^{2k})\Big] & \text{if } R_o \geq R_1, \\ \dfrac{(q+tk) + 2t(q+sk)r^k + (qt+s^2k)tr^{2k}}{(1+sr^k)(1+tr^k)} & \text{if } R_o \leq R_1 \end{cases}$$

where

$$R_o^2 = \frac{1+tkr^{k-1}-tkr^{k+1}-t^2r^{2k}}{qr^{k-1}(1-r^2) + 1-s^2r^{2k}}, \quad R_1 \equiv A-D = \frac{1+tr^k}{1+sr^k}, q \geq s, -1 \leq t < s \leq 1$$

and k is a fixed positive integer.

Proof: Let $p(z) = A+\xi+i\eta$ and $R^2=(A+\xi)^2+\eta^2$ with $|z| = r$. If we denote the left hand side of (3.2.8) by $U(\xi,\eta)$, then

$$(3.2.9)\quad U(\xi,\eta) = q(A+\xi) + tk(A+\xi)\, R^{-2} - \frac{1-s^2r^{2k}}{r^{k-1}(1-r^2)}(D^2-\xi^2-\eta^2)\, R^{-1}$$

and

$$(3.2.10)\quad \frac{\partial U}{\partial \eta} = \eta R^{-4}\, V(\xi,\eta)$$

where

$$
\begin{aligned}
V(\xi,\eta) &= -2tk(A+\xi)+\frac{1-s^2r^{2k}}{r^{k-1}(1-r^2)}(D^2-\xi^2-\eta^2)\,R+2\,\frac{1-s^2r^{2k}}{r^{k-1}(1-r^2)}\,R^3 \\
&\geq -2tk(A+\xi)+2\,\frac{1-s^2r^{2k}}{r^{k-1}(1-r^2)}\,R^3 \geq 2(A+\xi)\{-tk+\frac{1-s^2r^{2k}}{r^{k-1}(1-r^2)}\,R^2\} \\
&\geq 2(A+\xi)\{-tk+\frac{1-s^2r^{2k}}{r^{k-1}(1-r^2)}(A-D)^2\} = 2(A+\xi)\;\{-tk+\frac{(1+tr^k)^2(1-sr^k)}{r^{k-1}(1-r^2)(1+sr^k)}\} \\
&= \frac{2(A+\xi)}{r^{k-1}(1-r^2)(1+sr^k)}[(1-tkr^{k-1}+tkr^{k+1}+2tr^k+t^2r^{2k})+\overset{sr^k}{\cancel{s}}(-1-2tr^k \\
&\qquad\qquad -tkr^{k-1}+tkr^{k+1}-t^2r^{2k})] \\
&= \frac{2(A+\xi)}{r^{k-1}(1-r^2)(1+sr^k)}\;[\,J(s,t)\,]\quad (\text{say}).
\end{aligned}
$$

Since $J(s,t)$ decreases with s, it follows that

$$
\begin{aligned}
J(s,t) &\geq (1-r^k)(1+tr^k)^2-tkr^{k-1}(1-r^2)(1+r^k) \\
&\geq \frac{kr^{k-1}(1-r^2)}{1+r^k}\quad\{(1+tr^k)^2-t(1+r^k)^2\} \\
&= \frac{kr^{k-1}(1-r^2)(1-t)(1-tr^{2k})}{1+r^k} > 0.
\end{aligned}
$$

Hence

$$V(\xi,\eta) > 0$$

and so, by (3.2.10),

$$\frac{\partial U}{\partial \eta} = 0 \text{ if, and only if, } \eta = 0.$$

Therefore, the minimum of $U(\xi,\eta)$ on every chord ξ = constant is reached when $\eta = 0$ and thus the minimum of $U(\xi,\eta)$ in the circle $\xi^2+\eta^2 \leq D^2$ is attained on the diameter $\eta = 0$. On putting $\eta = 0$ in (3.2.9), we obtain

$$(3.2.11) \quad L(R) \equiv U(\xi, 0) = \left(q + \frac{1-s^2r^{2k}}{r^{k-1}(1-r^2)}\right)R + \frac{1+tkr^{k-1}-tkr^{k+1}-t^2r^{2k}}{r^{k-1}(1-r^2)} R^{-1} - 2A\frac{1-s^2r^{2k}}{r^{k-1}(1-r^2)}$$

where $R = A+\xi \in [A-D, A+D]$. It is easily seen that the absolute minimum of $L(R)$ in $(0,\infty)$ is attained at

$$(3.2.12) \qquad R_o = \left(\frac{1+tkr^{k-1}-tkr^{k+1}-t^2r^{2k}}{q(1-r^2)r^{k-1}+1-s^2r^{2k}}\right)^{1/2}$$

and the value of this minimum is

$$(3.2.13) \quad L(R_o) = \frac{2}{r^{k-1}(1-r^2)}\left[\sqrt{(qr^{k-1}(1-r^2)+1-s^2r^{2k})(1+tkr^{k-1}(1-r^2)-t^2r^{2k})} - (1-str^{2k})\right].$$

It is easy to check that $R_o <$ A+D, but is not always greater than A-D. In such a case when $R_o \notin [A-D, A+D]$, the minimum of $L(R)$ on the segment $[A-D, A+D]$ is attained at $R_1 = A-D$ since $L(R)$ increases with R on this segment. The value of this minimum equals

$$(3.2.14)\quad L(R_1) \equiv L(A\text{-}D) = \frac{(q+tk)+2t(q+sk)r^k+(qt+s^2k)tr^{2k}}{(1+sr^k)\,(1+tr^k)}.$$

Moreover $L(R_o) = L(R_1)$ for those values of q,k,s and t for which $R_o = R_1$. Hence the Lemma.

3.3 The radii of convexity for functions in $S^*(\alpha,\beta)$.

Theorem 3.3.1: Let $f \in S^*(\alpha,\beta)$ and let α_1(=0.335 approx) be the smallest positive root of the biquadratic equation

$$20\,\alpha^4 - 52\,\alpha^3 + 15\,\alpha^2 + 12\alpha - 4 = 0.$$

For a given $\beta(0<\beta\leq 1)$ let $\alpha_o(\beta)$ be the smallest positive root, lying in the interval $((3-\beta)/6,\ (5-3\beta)/5)$, of the 5th degree equation

$$Q(\alpha,\beta) \equiv 20\,\beta^4\alpha^5 - 4(17+\beta)\beta^3\alpha^4 + (35+32\beta)\beta^2\alpha^3 - (2-3\beta+4\beta^2)\,\beta\alpha^2 - (1+8\beta+7\beta^2)\,\alpha + (1+\beta)^2 = 0.$$

Further, let $V_1 = \{(\alpha,\beta):\ 0 \leq \alpha < 1,\ 0 < \beta \leq 1\}$, $A_1 = \{(\alpha,\beta): 0 \leq \alpha < \alpha_1, 0<\beta\leq 1\}$, $A_2 = \{(\alpha,\beta):\ \alpha_1 \leq \alpha \leq \alpha_o(\beta),\ 0 < \beta \leq 1\}$, $A_3 = V_1 - (A_1 \cup A_2)$. Then

(i) f is convex in $|z| < r_1$, for $(\alpha,\beta) \in A_1 \cup A_2$ and

(ii) f is convex in $|z| < r_2$, for $(\alpha,\beta) \in A_3$ where

$$r_1 = [(1+\beta-3\alpha\beta) + \sqrt{\beta(1-\alpha)(2+\beta-5\alpha\beta)}\,]^{-1}$$

and

$$r_2 = \left[\frac{5\alpha-1}{(1-\alpha+4\beta\alpha^2) + 4\alpha\sqrt{(1+\beta-3\alpha\beta+\alpha^2\beta^2)}}\right]^{1/2}.$$

s for $|z|$ in (i) and (ii) are sharp.

Proof: Since $f \in S^*(\alpha,\beta)$, we have by (2.2.7)

$$(3.3.1) \qquad z\frac{f'(z)}{f(z)} = \frac{1+(2\alpha\beta-1)z\phi(z)}{1+(2\beta-1)\,z\phi(z)}$$

where $\phi \in A$ for all $z \in \Delta$. Writing $z\phi(z) = \omega(z)$ where $\omega \in B$, we get

$$(3.3.2) \qquad z\frac{f'(z)}{f(z)} = \frac{1+(2\alpha\beta-1)\,\omega(z)}{1+(2\beta-1)\,\omega(z)}.$$

Differentiating (3.3.2) logarithmically, we have

$$(3.3.3) \qquad 1+z\frac{f''(z)}{f'(z)} = \frac{1+(2\alpha\beta-1)\omega(z)}{1+(2\beta-1)\,\omega(z)} - 2\beta(1-\alpha)\left\{\frac{z\omega'(z)}{(1+(2\alpha\beta-1)\omega(z))(1+(2\beta-1)\omega(z))}\right\}.$$

Applying (3.2.2) with $k = 1$, $s = 2\beta-1$, $t = 2\alpha\beta-1$ to (3.3.3), we get

$$(3.3.4) \qquad \mathrm{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} \geq \frac{1}{2\beta(1-\alpha)}\Big[\mathrm{Re}\left\{(2\beta(2-\alpha)-1)p(z) + \frac{2\alpha\beta-1}{p(z)}\right\} - \frac{r^2|(2\beta-1)p(z)-(2\alpha\beta-1)|^2-|1-p(z)|^2}{(1-r^2)|p(z)|}\Big] - \frac{\beta(1+\alpha)-1}{\beta(1-\alpha)}$$

where $p(z) = (1+(2\alpha\beta-1)\omega(z))/(1+(2\beta-1)\omega(z))$.

An application of Lemma 3.2.3 with $k = 1$, $q = 2\beta(2-\alpha)-1$, $s = 2\beta-1$, $t = 2\alpha\beta-1$ to (3.3.4) gives

$$(3.3.5)\quad \mathrm{Re}\{1+z\frac{f''(z)}{f'(z)}\} \geq \begin{cases} \frac{1}{\beta(1-\alpha)(1-r^2)}[\sqrt{4\alpha\beta^2(1-(2\alpha\beta-1)r^2)((2-\alpha)-(2\beta-\alpha)r^2)} \\ \quad - (1+(1-2\alpha\beta)(2\beta-1)r^2)+(1-\beta-\alpha\beta)(1-r^2)] & \text{if } R_o \geq R_1, \\ \frac{1-2(1+\beta-3\alpha\beta)r+(1-2\alpha\beta)^2r^2}{(1+(2\beta-1)r)(1+(2\alpha\beta-1)r)} & \text{if } R_o \leq R_1 \end{cases}$$

where

$$R_o^2 = \frac{\alpha(1-(2\alpha\beta-1)r^2)}{[(2-\alpha)-(2\beta-\alpha)r^2]}, \quad R_1 = \frac{1+(2\alpha\beta-1)r}{1+(2\beta-1)r}.$$

By (3.3.5) the bound r' of convexity for the functions of the class $S^*(\alpha,\beta)$ is determined either from the equation

$$(3.3.6)\qquad (1-2\alpha\beta)^2r^2-2\,(1+\beta-3\alpha\beta)r + 1 = 0,$$

or from the equation

$$(3.3.7)\qquad 2\beta\sqrt{\alpha(1-(2\alpha\beta-1)r^2)((2-\alpha)-(2\beta-\alpha)r^2)} - (1+(1-2\alpha\beta)(2\beta-1)r^2) + (1-\beta-\alpha\beta)(1-r^2) = 0.$$

Equation (3.3.7) reduces to

$$(3.3.8)\qquad (8\beta\alpha^2-3\alpha-1)r^4-2(4\beta\alpha^2-\alpha+1)r^2+ (5\alpha-1) = 0.$$

Also the minima given by (3.3.5) become equal to each other for those $(\alpha,\beta) \in V_1$ for which

$$(3.3.9)\qquad R_o = R_1.$$

From (3.3.6) and (3.3.8), we obtain

$$(3.3.10) \qquad r' \equiv r_1 = [(1+\beta-3\alpha\beta) + \sqrt{\beta(1-\alpha)(2+\beta-5\alpha\beta)}]^{-1}$$

and

$$(3.3.11) \qquad r' \equiv r_2 = \left[\frac{5\alpha-1}{(1-\alpha+4\beta\alpha^2) + 4\alpha\sqrt{(1+\beta-3\alpha\beta+\alpha^2\beta^2)}}\right]^{1/2}.$$

To obtain the points $(\alpha,\beta) \in V_1$ which determine the transition from formula (3.3.10) to formula (3.3.11) we eliminate r from (3.3.6) and (3.3.9) and get

$$(3.3.12) \qquad Q(\alpha,\beta) \equiv 20\beta^4\alpha^5 - 4(17+\beta)\beta^3\alpha^4 + (35+32\beta)\beta^2\alpha^3 - (2-3\beta+4\beta^2)\beta\alpha^2$$
$$-(1+8\beta+7\beta^2)\alpha + (1+\beta)^2 = 0.$$

For a given $\beta(0 < \beta \le 1)$, let $\alpha_0(\beta)$ be the smallest positive root of the equation (3.3.12). It is easy to check that $\alpha_0(\beta)$ lies in the interval $((3-\beta)/6, (5-3\beta)/5)$.

For $\beta = 1$, $\alpha_0(1) = \alpha_1 (=0.335$ approx) is the smallest positive root, lying in the interval $(1/3, 2/5)$ of the biquadratic equation

$$20\alpha^4 - 52\alpha^3 + 15\alpha^2 + 12\alpha - 4 = 0$$

and $\alpha = 1$ as β tends to zero.

Let C denote the arc of the curve $Q(\alpha,\beta) = 0$ lying in the region $X = \{(\alpha,\beta): \alpha_1 \le \alpha < 1, 0 < \beta \le 1\} \subset V_1$ i.e. passing through the points $(\alpha_1, 1)$ and $(1,0)$. The curve C divides the region V_1 into two subregions $Y = A_1 \cup A_2$ and A_3 where $A_1 = \{(\alpha,\beta): 0 \le \alpha < \alpha_1, 0 < \beta \le 1\}$, $A_2 = \{(\alpha,\beta): \alpha_1 \le \alpha \le \alpha_0(\beta), 0 < \beta \le 1\}$ and $A_3 = V_1 - (A_1 \cup A_2)$. The curve C also gives transition from formula (3.3.10) ~~xxxxxxxxxx~~

to formula (3.3.11). It is obvious that C has a void intersection with A_1 so that in A_1 we have to use either formula (3.3.10) or formula (3.3.11). But it is easily seen that it is impossible to use formula (3.3.11) for all (α,β) lying in $Y' = \{(\alpha,\beta): 0 \leq \alpha \leq 1/5,$ $0 < \beta \leq 1\} \subset A_1$. So formula (3.3.10) must be used for (α,β) lying in A_1. Now we consider the points (α,β) lying in the region $X = A_2 \cup A_3$. Since formula (3.3.10) cannot be used for points (α,β) lying in $W = \{(\alpha,\beta): \alpha_0'(\beta) \leq \alpha < 1,\ 1/2 \leq \beta \leq 1,\ \alpha_0'(\beta) = (2+\beta)/5\beta\} \subset A_3$, it therefore follows that for $(\alpha,\beta) \in A_3$, we have to use formula (3.3.11) while formula (3.3.10) is to be used for $(\alpha,\beta) \in A_1 \cup A_2$. This proves (i) and (ii).

Figure 2 shows the transition curve C and the regions A_1, A_2 and A_3.

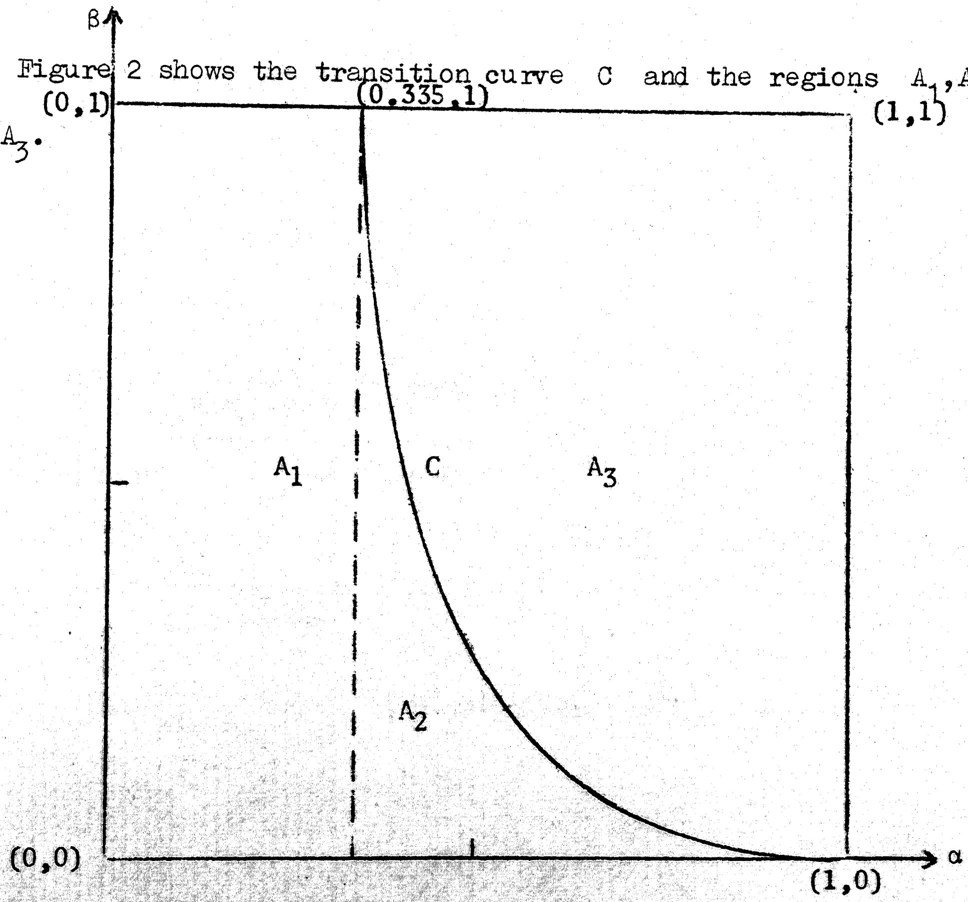

FIG. 2

The function f given by

$$(3.3.13) \qquad z\frac{f'(z)}{f(z)} = \frac{1-(2\alpha\beta-1)z}{1-(2\beta-1)z}$$

shows that f is not convex in any circle $|z| < r$ if $r > r_1$ in estimate (i) since

$$1+z\frac{f''(z)}{f'(z)} = \frac{1+2(1+\beta-3\alpha\beta)z + (2\alpha\beta-1)^2z^2}{(1-(2\beta-1)z)(1-(2\alpha\beta-1)z)} = 0$$

for $z = -r_1$. To see that the estimate (ii) is sharp, consider the function f given by

$$(3.3.14) \qquad z\frac{f'(z)}{f(z)} = \frac{1-2\alpha\beta bz + (2\alpha\beta-1)z^2}{1-2\beta bz + (2\beta-1)z^2}$$

where b is determined by the relation

$$(3.1.15) \qquad \frac{1-2\alpha\beta br + (2\alpha\beta-1)r^2}{1-2\beta br + (2\beta-1)r^2} = R_o = \left(\frac{\alpha(1-(2\alpha\beta-1)r^2)}{(2-\alpha)-(2\beta-\alpha)r^2}\right)^{1/2}.$$

Differentiating (3.3.14) logarithmically, we have

$$1+z\frac{f''(z)}{f'(z)}\bigg|_{z=r} = \frac{1+2\beta(1-3\alpha)br+(4\alpha^2\beta^2b^2-2-4\beta+4\alpha\beta)r^2+(2\beta+2\alpha\beta-8\alpha^2\beta^2)br^3+(2\alpha\beta-1)^2r^4}{(1-2\beta br+(2\beta-1)r^2)(1-2\alpha\beta br+(2\alpha\beta-1)r^2)}$$

$$= \frac{1}{\beta(1-\alpha)(1-r^2)}\,[((2-\alpha)\beta-(2\beta^2-\alpha\beta)r^2)R_o + \frac{\alpha\beta(1-(2\alpha\beta-1)r^2)}{R_o} -$$

$$(1+(1-2\alpha\beta)(2\beta-1)r^2) + (1-\beta-\alpha\beta)(1-r^2)]$$

$$= \frac{1}{\beta(1-\alpha)(1-r^2)}\left[\sqrt{4\alpha\beta^2(1-(2\alpha\beta-1)r^2)((2-\alpha)-(2\beta-\alpha)r^2)}-(1+(1-2\alpha\beta)(2\beta-1)r^2)\right.$$

$$\left. + (1-\beta-\alpha\beta)(1-r^2)\right]$$

$$= 0$$

for $r = r_2$. Hence the function f in estimate (ii) is, therefore, not convex in any circle $|z| < r$ if $r > r_2$. This completes the proof of the Theorem.

Corollary 3.3.1a [74,78,84]: Let $f \in S^*_\alpha$ and let α_1(=0.335 approx) be the smallest positive root, lying in the interval (1/3, 2/5) of the biquadratic equation

$$20\alpha^4-52\alpha^3+15\alpha^2+12\alpha-4 = 0.$$

Then

(i) for $0 \leq \alpha \leq \alpha_1$, f is convex in

$$|z| < [(2-3\alpha) + \sqrt{(1-\alpha)(3-5\alpha)}]^{-1} ,$$

(ii) for $\alpha_1 \leq \alpha < 1$, f is convex in

$$|z| < \left[\frac{5\alpha-1}{(1-\alpha+4\alpha^2) + 4\alpha\sqrt{(1-\alpha)(2-\alpha)}}\right]^{1/2} .$$

The estimates for $|z|$ in (i) and (ii) are both sharp for the functions given by (3.3.13) and (3.3.14) with $\beta = 1$.

in the field by using different techniques. It also gives results due to Schild [73], MacGregor [43] etc. for $\alpha = 1/2$.

Corollary 3.3.1b [61]: Let $f \in S^*(\gamma)$, $0 < \gamma \leq 1$. Then

(i) f maps $|z| < (2-\sqrt{3})/\gamma$ onto a convex domain if

$$(2\sqrt{3} - 3) \leq \gamma \leq 1,$$

(ii) f maps

$$|z| < [\{-(1-\gamma+\gamma^2) + (1-\gamma)\sqrt{(\gamma^2+6\gamma+1)}\}/(3\gamma-2\gamma^2)]^{1/2}$$

onto a convex domain if

$$0 < \gamma \leq (2\sqrt{3}-3).$$

The bounds for $|z|$ in (i) and (ii) are sharp.

This result is obtained by replacing α by $(1-\gamma)/(1+\gamma)$ and β by $(1+\gamma)/2$ in Theorem 3.3.1. The above result was obtained by Padmanabhan [61] by using a different technique.

Also, replacing α by $1-\alpha$ and β by $1/2$ in Theorem 3.3.1, we get the following result due to Eenigenburg [13].

Corollary 3.3.1c: Let $f \in \bar{S}^*(\alpha)$ and let $\alpha_0 = \{(3-\sqrt{5}) + 2\sqrt{3(7-3\sqrt{5})}\}/2\sqrt{5}$ ≈ 0.589. Then

(i) for $\alpha_0 \leq \alpha \leq 1$, f is convex in

$$|z| < (3-\sqrt{5})/2\alpha,$$

(ii) <u>for</u> $0 < \alpha \leq \alpha_0$, f <u>is convex in</u>

$$|z| < [\{-(2\alpha^2-3\alpha+2) + 2(1-\alpha)\sqrt{(\alpha^2+4\alpha+1)}\}/(5\alpha-4\alpha^2)]^{1/2} .$$

<u>The bounds for</u> $|z|$ <u>in</u> (i) <u>and</u> (ii) <u>are sharp</u>

<u>Corollary 3.3.1d</u> [53]: <u>Let</u> $f \in \bar{S}^*(1-\alpha)$ <u>and let</u> $\alpha_0'=1-(1+\sqrt{6})(3\sqrt{5}-5)/10$ $\approx 0.411....$ <u>Then</u>

(i) <u>for</u> $0 \leq \alpha \leq \alpha_0'$, f <u>is convex in</u>

$$|z| < (3-\sqrt{5})/2(1-\alpha),$$

(ii) <u>for</u> $\alpha_0' \leq \alpha < 1$, f <u>is convex in</u>

$$|z| < [\{(-2\alpha^2+\alpha-1) + 2\alpha\sqrt{(6-6\alpha+\alpha^2)}\}/(1+3\alpha-4\alpha^2)]^{1/2} .$$

<u>The bounds for</u> $|z|$ <u>in</u> (i) <u>and</u> (ii) <u>are sharp</u>.

The above result is obtained by taking $\beta = 1/2$ in Theorem 3.3.1 or by replacing α by $1-\alpha$ in Corollary 3.3.1c.

Putting $(\alpha,\beta) = (0,(2\delta-1)/2\delta)$ and $(\alpha,\beta) = (0,1/2)$ in Theorem 3.3.1, we get respectively the following results obtained by Ram Singh [68,69] and Singh and Goel [78].

<u>Corollary 3.3.1e</u>: <u>Each function</u> f(z) in $\bar{S}(\delta)$ <u>maps</u> $|z| < [\{(4\delta-1) + \sqrt{(2\delta-1)(6\delta-1)}\}/2\delta]^{-1}$ <u>onto a convex domain</u>. <u>The result is sharp</u>.

<u>Corollary 3.3.1f</u>: <u>Each function</u> $f \in \bar{S}^*$ <u>maps</u> $|z| < (3-\sqrt{5})/2$ <u>onto a convex domain</u>. <u>The result is sharp</u>.

<u>Remark 3.3.1</u>: Substracting γ from both sides of the equation (3.3.5) and proceeding on the similar lines as in the above theorem, we may get radii of convexity of order γ for functions in $S^*(\alpha,\beta)$. The result so obtained generalizes the corresponding result for the class S^*_α recently obtained by Bajpai [3] .

We now obtain the radii of convexity for functions belonging to the class $S^*_k(\alpha,\beta)$.

<u>Theorem 3.3.2</u>: <u>Let</u> $f \in S^*_k(\alpha,\beta)$ <u>and let</u> $r'(\alpha,\beta,k)$ <u>be the smallest positive root of the equation</u>

$$\{(1-2\alpha\beta)(2\beta(k-\alpha)-k+1)r^{2k}-2(1-2\alpha\beta)r^{k}+(k+1)\}(1-r^2)$$
$$-2r(1+(2\beta-1)r^{k})(1-(1-2\alpha\beta)r^{k}) = 0.$$

<u>Then</u>

(i) <u>for</u> $0 < r \leq r'(\alpha,\beta,k)$, f <u>is convex in</u>

$$|z| < r_1 \equiv [(1-2\alpha\beta+k(1-\alpha)\beta) + \sqrt{\beta k(1-\alpha)(\beta k(1-\alpha)+2(1-2\alpha\beta))}\,]^{-1/k}$$

(ii) <u>for</u> $r'(\alpha,\beta,k) \leq r < 1$, f <u>is convex in</u>

$$|z| < r_2,$$

<u>where</u> r_2 <u>is the smallest positive root of the equation</u>

$$2(1-2\alpha\beta)^2r^{2k}(1-r^2)+k(k\beta(1-\alpha)+2(1-2\alpha\beta))r^{k+3}+2(\beta(1-\alpha)(2-k^2)-2k(1-2\alpha\beta))$$
$$r^{k+1}+k\,(\beta(1-\alpha)k+2(1-2\alpha\beta))r^{k-1}+2r^2-2 = 0.$$

<u>The bounds for $|z|$ in (i) and (ii) are sharp</u>.

__Proof__ : Since $f \in S_k^*(\alpha,\beta)$, we have by (2.2.5)

$$(3.3.16)\qquad z\frac{f'(z)}{f(z)} = \frac{1+(2\alpha\beta-1)z^k\phi(z)}{1+(2\beta-1)z^k\phi(z)}$$

where $\phi \in A$ for all $z \in \Delta$. Writing $z\phi(z) = \omega(z)$ where $\omega \in \mathcal{B}$, we get

$$(3.3.17)\qquad z\frac{f'(z)}{f(z)} = \frac{1+(2\alpha\beta-1)z^{k-1}\omega(z)}{1+(2\beta-1)z^{k-1}\omega(z)}\,.$$

Differentiating (3.3.17) logarithmically, we have

$$(3.3.18)\qquad 1+z\frac{f''(z)}{f'(z)} = \frac{1+(2\alpha\beta-1)z^{k-1}\omega(z)}{1+(2\beta-1)z^{k-1}\omega(z)} - 2\beta(1-\alpha)\left\{\frac{z^k\omega'(z)+(k-1)\,z^{k-1}\omega(z)}{(1+(2\beta-1)z^{k-1}\omega(z))(1+(2\alpha\beta-1)\,z^{k-1}\omega(z))}\right\}.$$

Applying (3.2.2) with $s = 2\beta-1$, $t = 2\alpha\beta-1$ to (3.3.18), we obtain

$$(3.3.19)\qquad \mathrm{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} \geq \frac{1}{2\beta(1-\alpha)}\left[\mathrm{Re}\left\{((2\beta-1)k+2\beta(1-\alpha))p(z)+\frac{(2\alpha\beta-1)k}{p(z)}\right\} - \frac{r^{2k}|(2\beta-1)p(z)-(2\alpha\beta-1)|^2-|1-p(z)|^2}{r^{k-1}\,(1-r^2)\,|p(z)|}\right] + \frac{k(1-\beta-\alpha\beta)}{\beta(1-\alpha)}$$

where $p(z) = (1+(2\alpha\beta-1)z^{k-1}\omega(z))/(1+(2\beta-1)z^{k-1}\omega(z))$.

An application of Lemma 3.2.3 with $q = (2\beta-1)k+2\beta(1-\alpha)$, $s=2\beta-1$, $t = 2\alpha\beta-1$ to (3.3.19) gives

$$(3.3.20)\quad \mathrm{Re}\ \{1+z\frac{f''(z)}{f'(z)}\} \geq \begin{cases} \dfrac{1}{\beta(1-\alpha)r^{k-1}(1-r^2)}[\sqrt{M}-(1+(2\beta-1)(1-2\alpha\beta)r^{2k} \\ \qquad + kr^{k-1}(1-r^2)(1-\beta-\alpha\beta)] & \text{if } R_o \geq R_1, \\ \dfrac{1-2((1-2\alpha\beta)+\beta k(1-\alpha))r^k+(1-2\alpha\beta)^2r^{2k}}{(1+(2\beta-1)r^k)(1+(2\alpha\beta-1)r^k)} & \text{if } R_o \leq R_1 \end{cases}$$

where M stands for

$$(3.3.20;a)\ M = \{((2\beta-1)k+2\beta(1-\alpha))r^{k-1}(1-r^2)+1-(2\beta-1)^2r^{2k}\}\{1+k(2\alpha\beta-1)r^{k-1}(1-r^2) \\ -(1-2\alpha\beta)^2r^{2k}\}$$

and

$$R_o^2 = \frac{1+k(2\alpha\beta-1)\ r^{k-1}(1-r^2)-(1-2\alpha\beta)^2r^{2k}}{((2\beta-1)k+2\beta(1-\alpha))r^{k-1}(1-r^2)+1-(2\beta-1)^2r^{2k}},\ R_1 = \frac{1+(2\alpha\beta-1)r^k}{1+(2\beta-1)r^k}.$$

Right hand sides of both the inequalities in (3.3.20) become equal to each other for those values of α, β and k for which

$$(3.3.21)\qquad R_o = R_1.$$

The equation (3.3.21) on simplification yields

$$(3.3.22)\qquad \{(1-2\alpha\beta)(2\beta(k-\alpha)-k+1)r^{2k}-2(1-2\alpha\beta)r^k+(k+1)\}(1-r^2) \\ -2r(1+(2\beta-1)r^k)(1-(1-2\alpha\beta)r^k) = 0.$$

Let $r'(\alpha,\beta,k)$ be the smallest positive root of the equation (3.3.22). Then (3.3.20) is equivalent to

$$(3.3.23)\quad \mathrm{Re}\{1+z\frac{f''(z)}{f'(z)}\} \geq \begin{cases} \dfrac{1}{\beta(1-\alpha)r^{k-1}(1-r^2)}[\sqrt{M}-(1+(2\beta-1)(1-2\alpha\beta)r^{2k}) \\ \qquad + kr^{k-1}(1-r^2)(1-\beta-\alpha\beta)] \\ \qquad\qquad \text{if } r'(\alpha,\beta,k) \leq r < 1, \\ \dfrac{1-2((1-2\alpha\beta)+\beta k(1-\alpha))r^k+(1-2\alpha\beta)^2r^{2k}}{(1+(2\beta-1)r^k)(1+(2\alpha\beta-1)r^k)} \\ \qquad\qquad \text{if } 0 < r \leq r'(\alpha,\beta,k) \end{cases}$$

where M is given by (3.3.20;a).

Thus we see that the radii of convexity r_1 and r_2 for functions in $S_k^*(\alpha,\beta)$ are respectively given by the smallest positive root of the equations

$$(3.3.24)\qquad 1-2((1-2\alpha\beta)+\beta k(1-\alpha))r^k+(1-2\alpha\beta)^2r^{2k}=0$$

$$\text{if } 0 < r \leq r'(\alpha,\beta,k),$$

$$(3.3.25)\qquad \sqrt{M}-(1+(2\beta-1)(1-2\alpha\beta)r^{2k})+kr^{k-1}(1-r^2)(1-\beta-\alpha\beta)=0$$

$$\text{if } r'(\alpha,\beta,k) \leq r < 1$$

where M is given by (3.3.20;a).

The equation (3.3.25) may be reduced to

$$(3.3.26)\qquad 2(1-2\alpha\beta)^2r^{2k}(1-r^2)+k(\beta(1-\alpha)k+2(1-2\alpha\beta))r^{k+3}+2(\beta(1-\alpha)(2-k^2)-$$
$$2k(1-2\alpha\beta))r^{k+1}+k(\beta(1-\alpha)k+2(1-2\alpha\beta))r^{k-1}+2r^2-2=0 \text{ if}$$
$$r'(\alpha,\beta,k) \leq r < 1.$$

It is easily seen that the smallest positive root of the equation (3.3.24) is given by

$$(3.3.27)\qquad r_1=[(1-2\alpha\beta+k\beta(1-\alpha))+\sqrt{\beta k(1-\alpha)(\beta k(1-\alpha)+2(1-2\alpha\beta))}]^{-1/k}$$

if $0 < r \leq r'(\alpha,\beta,k)$. This proves (i) and (ii).

Functions given by

$$z\frac{f'(z)}{f(z)} = \frac{1-(2\alpha\beta-1)z^k}{1-(2\beta-1)\ z^k} \tag{3.3.27}$$

and

$$z\frac{f'(z)}{f(z)} = \frac{1-(1+(2\alpha\beta-1)z^{k-1})bz + (2\alpha\beta-1)z^{k+1}}{1-(1+(2\beta-1)\ z^{k-1})bz + (2\beta-1)z^{k+1}} \tag{3.3.28}$$

where b is determined by the relation

$$\frac{1-(1+(2\alpha\beta-1)\ r^{k-1})br + (2\alpha\beta-1)r^{k+1}}{1-(1+(2\beta-1)r^{k-1})br + (2\beta-1)\ r^{k+1}} = \sqrt{\frac{1+(2\alpha\beta-1)kr^{k-1}(1-r^2)-(2\alpha\beta-1)^2r^{2k}}{1+((2\beta-1)k+2\beta(1-\alpha))r^{k-1}(1-r^2)-(2\beta-1)^2r^{2k}}}$$

$$= R_o$$

show that the results obtained in (i) and (ii) are sharp.

Corollary 3.3.2a : Let $f \in S_k^*(\gamma)$ and let $\gamma_o = \{(k+1)-\sqrt{(k+1)^2-1}\}\{\frac{k+2}{\sqrt{(k+1)^2-1}}\}^k$.

Then

(i) for $\gamma_o \leq \gamma \leq 1$, f is convex in

$$|z| < r_1 \equiv [\gamma\{(k+1) + \sqrt{(k+1)^2-1}\}]^{-1/k} ,$$

(ii) for $0 < \gamma \leq \gamma_o$, f is convex in

$$|z| < r_2,$$

where r_2 is the smallest positive root of the equation

$$2\gamma^2 r^{2k}(1-r^2)+\gamma k(2+k)r^{k+3}+2\gamma(2-k(2+k))r^{k+1}+\gamma k(2+k)r^{k-1}+2r^2-2=0.$$

The bounds for $|z|$ in (i) and (ii) are both sharp.

Proof : The result follows by replacing α by $(1-\gamma)/(1+\gamma)$ and β by $(1+\gamma)/2$ in equation (3.3.20) and proceeding on the similar lines as in Theorem 3.3.1. The sharpness is given by (3.3.27) and (3.3.28) with $\alpha=(1-\gamma)/(1+\gamma)$ and $\beta=(1+\gamma)/2$.

Putting $\beta = 1$, in Theorem 3.3.2, we deduce the following result for the functions in $S^*_{\alpha,k}$.

Corollary 3.3.2b: Let $f \in S^*_{\alpha,k}$ and let $r'(\alpha,k)$ be the smallest positive root of the equation

$$\{(1-2\alpha)(2(k-\alpha)-k+1)r^{2k}-2(1-2\alpha)r^k+(k+1)\}(1-r^2)-2r(1+r^k)(1-(1-2\alpha)r^k)=0.$$

Then

(i) for $0 < r \leq r'(\alpha, k)$, f is convex in

$$|z| < r_1 \equiv [(1-2\alpha+k(1-\alpha))+\sqrt{k(1-\alpha)(k(1-\alpha)+2(1-2\alpha))}]^{-1/k},$$

(ii) for $r'(\alpha,k) \leq r < 1$, f is convex in

$$|z| < r_2,$$

where r_2 is the smallest positive root of the equation

$$2(1-2\alpha)^2 r^{2k}(1-r^2)+k(k(1-\alpha)+2(1-2\alpha))r^{k+3}+2((1-\alpha)(2-k^2)-2k(1-2\alpha))r^{k+1}$$

$$+k((1-\alpha)k+2(1-2\alpha))r^{k-1}+2r^2-2=0.$$

The bounds for $|z|$ *in* (i) *and* (ii) *are sharp.*

For $\beta = 1/2$, in Theorem 3.3.2 we get the following result for the functions in $\bar{S}_k^*(1-\alpha)$.

Corollary 3.3.2c: Let $f \in \bar{S}_k^*(1-\alpha)$ *and let* $r_o(\alpha,k)$ *be the smallest positive root of the equation*

$$\{(1-\alpha)^2 r^{2k} - 2(1-\alpha)r^k + (k+1)\}(1-r^2) - 2r(1-(1-\alpha)r^k) = 0.$$

Then

(i) *for* $0 < r \leq r_o(\alpha,k)$, f *is convex in*

$$|z| < r_1 \equiv [1/2(1-\alpha)\{(2+k) + \sqrt{k(4+k)}\}]^{-1/k},$$

(ii) *for* $r_o(\alpha,k) \leq r < 1$, f *is convex in*

$$|z| < r_2,$$

where r_2 *is the smallest positive root of the equation*

$$4(1-\alpha)^2 r^{2k}(1-r^2) + k(k+4)(1-\alpha)r^{k+3} + 2(1-\alpha)(2-4k-k^2) r^{k+1} +$$
$$k(4+k)(1-\alpha)r^{k-1} + 4r^2 - 4 = 0.$$

The bounds for $|z|$ *in* (i) *and* (ii) *are sharp.*

Remark 3.3.2: (a) Replacing α by $1-\alpha$ and β by $1/2$ in Theorem 3.3.2, we get the corresponding result for functions in $\bar{S}_k^*(\alpha)$.

(b) $(\alpha,\beta) = (0,(2\delta-1)/2\delta)$ and $(\alpha,\beta) = (0, 1/2)$ in Theorem 3.3.2 leads respectively to the corresponding results for functions in $\bar{S}_k(\delta)$ and $\bar{S}_k^*$.

3.4 The radii of starlikeness for functions in $V^*(\alpha,\beta)$.

Livingston [39] proved that if $f \in S^*$, then $1/2(zf(z))'$ belongs to S^* for $|z| < 1/2$. Singh and Goel [78], Libera and Livingston [37], Bajpai and Singh [4], Al-Amiri [2] etc. extended this result to the case of functions of the class S^*_α. In this section, we determine the exact bounds for the radii of starlikeness for the functions of the form (3.1.1) where $f \in S^*(\alpha,\beta)$. For different values of the parameters α,β $(0 \le \alpha < 1,\ 0 < \beta \le 1)$, our results in this section yield the corresponding results for the classes $V^*(\gamma)$, V^*_α, $\bar{V}^*(\alpha)$, $\bar{V}^*(1-\alpha)$, $\bar{V}(\delta)$ and $\bar{V}^*$ of functions of the form (3.1.1) where f belongs respectively to the classes $S^*(\gamma)$, S^*_α, $\bar{S}^*(\alpha)$, $\bar{S}^*(1-\alpha)$, $\bar{S}(\delta)$ and $\bar{S}^*$. Thus we have the following result:

Theorem 3.4.1: Let $F \in V^*(\alpha,\beta)$ and let $\alpha_0 = 1/3$. For a given $\beta\,(0 < \beta \le 1)$ let $\alpha_0(\beta)$ be the smallest positive root, lying in the interval $((4-3\beta)/20,\ (5-2\beta)/15)$, of the 4th degree equation

$$\bar{Q}(\alpha,\beta) \equiv 4\beta^3\alpha^4 - 8(2-\beta)\beta^2\alpha^3 + \beta(4\beta^2-16\beta+6)\alpha^2 + (8\beta+3)\alpha + 1 = 0.$$

Further, let $V_1 = \{(\alpha,\beta) : 0 \le \alpha < 1,\ 0 < \beta \le 1\}$, $B_1 = \{(\alpha,\beta) : 0 \le \alpha \le \alpha_0(\beta),\ 0 < \beta \le 1\}$, $B_2 = \{(\alpha,\beta) : \alpha_0(\beta) < \alpha \le \alpha_0,\ 0 < \beta \le 1\}$, $B_3 = V_1 - (B_1 \cup B_2)$.

Then

(i) F is starlike in $|z| < r_1$, for $(\alpha,\beta) \in B_1$ and

(ii) F is starlike in $|z| < r_2$, for $(\alpha,\beta) \in B_2 \cup B_3$ where

$$r_1 = [\,(1-2\alpha\beta) + \sqrt{\beta(1-\alpha)(1-2\alpha\beta)}\,]^{-1}$$

and

$$r_2 = \left[\frac{2\alpha}{\alpha\beta(1+\alpha)+\sqrt{(2\alpha+(2-4\beta+\beta^2)\alpha^2-2\beta(2-\beta)\alpha^3+\beta^2\alpha^4)}}\right]^{1/2}$$

The bounds for $|z|$ in (i) and (ii) are sharp.

Proof: Since $f \in S^*(\alpha,\beta)$, we have by (3.3.2)

$$z\frac{f'(z)}{f(z)} = \frac{1+(2\alpha\beta-1)\,\omega(z)}{1+(2\beta-1)\,\omega(z)} \tag{3.4.1}$$

where $\omega \in \mathcal{B}$. Differentiating (3.1.1) and using (3.4.1), we obtain

$$z\frac{F'(z)}{F(z)} = \frac{1+(2\alpha\beta-1)\,\omega(z)}{1+(2\beta-1)\,\omega(z)} - (1-\alpha)\beta\left\{\frac{z\omega'(z)}{(1+(2\beta-1)\omega(z))(1+(\alpha\beta+\beta-1)\omega(z))}\right\} \tag{3.4.2}$$

Applying (3.2.2) with $k = 1$, $s = 2\beta-1$, $t = \alpha\beta+\beta-1$ to (3.4.2), we get

$$\operatorname{Re}\left\{z\frac{F'(z)}{F(z)}\right\} \geq \frac{1}{\beta(1-\alpha)}\left[\operatorname{Re}\left\{(4\beta-1-2\alpha\beta)p(z)+\frac{\alpha\beta+\beta-1}{p(z)}\right\} - \frac{r^2|(2\beta-1)p(z)-(\alpha\beta+\beta-1)|^2-|1-p(z)|^2}{(1-r^2)\,|p(z)|}\right] - \frac{2(2\beta-1)}{\beta(1-\alpha)}. \tag{3.4.3}$$

where $p(z) = (1+(\alpha\beta+\beta-1)\,\omega(z))/(1+(2\beta-1)\,\omega(z))$.

An application of Lemma 3.2.3 with $k = 1$, $q = 4\beta-1-2\alpha\beta$, $s = 2\beta-1$, $t = \alpha\beta+\beta-1$ to (3.4.3) gives

$$\operatorname{Re}\left\{z\frac{F'(z)}{F(z)}\right\} \geq \begin{cases} \dfrac{2}{\beta(1-\alpha)(1-r^2)}\Big[\sqrt{2\beta^2(1+\alpha)[(2-\alpha)-(2\beta-\alpha)r^2][1-(\alpha\beta+\beta-1)r^2]} \\ \qquad - (1-(2\beta-1)(\alpha\beta+\beta-1)r^2)-(2\beta-1)(1-r^2)\Big] \\ \qquad\qquad \text{if } R_0 \geq R_1, \\ \dfrac{1-2(1-2\alpha\beta)r+(\alpha\beta+\beta-1)(2\alpha\beta-1)r^2}{(1+(2\beta-1)r)(1+(\alpha\beta+\beta-1)r)} \\ \qquad\qquad \text{if } R_0 \leq R_1 \end{cases} \tag{3.4.4}$$

where

$$R_o^2 = \frac{1}{2}\ \frac{(1+\alpha)\ [1-(\alpha\beta+\beta-1)r^2]}{(2-\alpha)-(2\beta-\alpha)r^2}\ ,\ R_1 = \frac{1+(\alpha\beta+\beta-1)r}{1+(2\beta-1)r}.$$

Proceeding on similar lines as in Theorem 3.3.1, we get the required results easily.

Figure 3 shows the transition curve $\bar{C}$ given by $\bar{Q}(\alpha,\beta)=0$ and the regions B_1, B_2 and B_3.

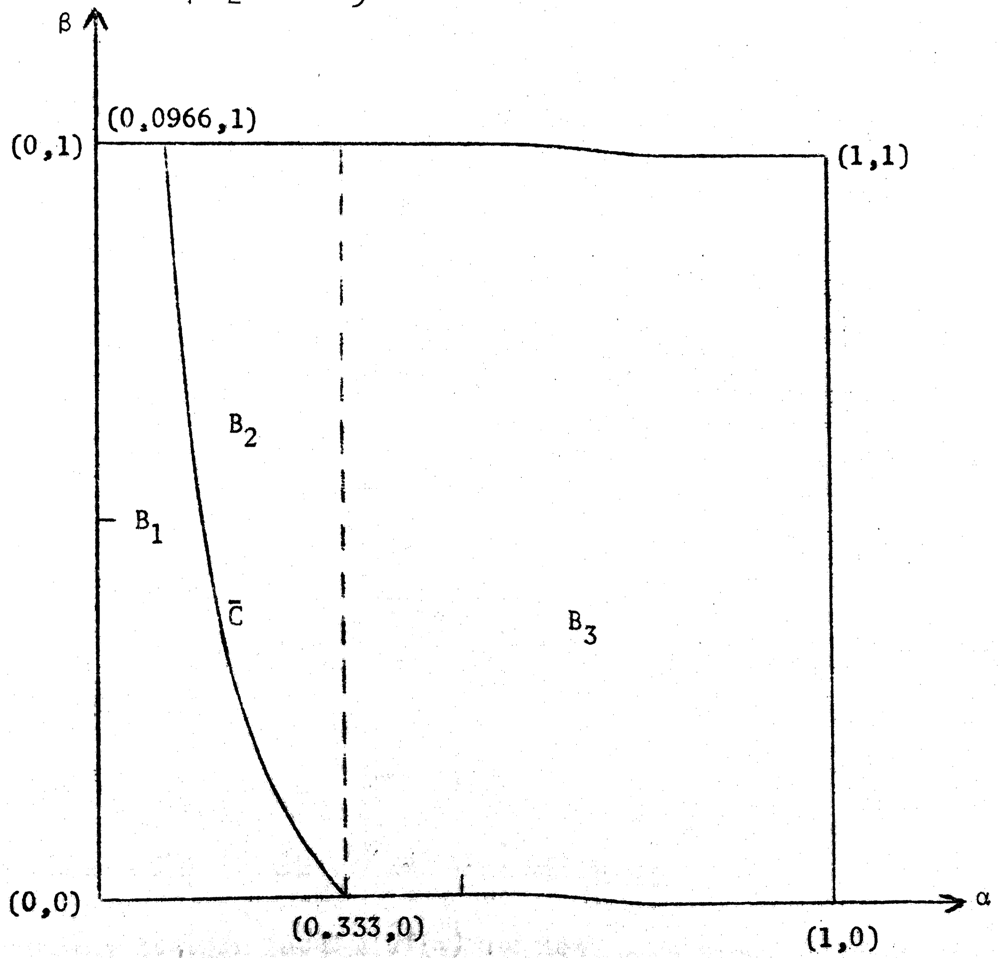

FIG. 3

Functions given by

$$(3.4.5)\qquad z\frac{f'(z)}{f(z)} = \frac{1-(2\alpha\beta-1)z}{1-(2\beta-1)z}$$

and

$$(3.4.6) \qquad z\frac{f'(z)}{f(z)} = \frac{1-2\alpha\beta bz + (2\alpha\beta-1)z^2}{1-2\beta bz + (2\beta-1)z^2}$$

where b is determined by the relation

$$\frac{1-\beta(1+\alpha)br + (\alpha\beta+\beta-1)r^2}{1-2\beta br + (2\beta-1)r^2} = \sqrt{\frac{\beta(1+\alpha)(1-(\alpha\beta+\beta-1)r^2)}{(4\beta-1-2\alpha\beta)(1-r^2)+1-(2\beta-1)^2r^2}} = R_o$$

show that the results obtained in (i) and (ii) are sharp.

Replacing α by $(1-\gamma)/(1+\gamma)$ and β by $(1+\gamma)/2$ in Theorem 3.4.1, we deduce the following result:

Corollary 3.4.1a: *If* $F \in V^*(\gamma)$, $0 < \gamma \leq 1$, *then*

(i) F *maps* $|z| < 1/2\gamma$ *onto a starlike domain if*

$$(1+\sqrt{5})/4 \leq \gamma \leq 1,$$

(ii) F *maps*

$$|z| \leq \left[\frac{2(1-\gamma)}{(1-\gamma)+\sqrt{(1-\gamma)(1+3\gamma)}}\right]^{1/2}$$

onto a starlike domain if

$$0 < \gamma \leq (1+\sqrt{5})/4 \ .$$

The bounds for $|z|$ *in* (i) *and* (ii) *are sharp*.

Corollary 3.4.1b: *Let* $F \in \bar{V}^*(\alpha)$ *and let* α_o (= 0.855...approx) *be the smallest positive root of the equation*

$$\alpha^4 + 2\alpha^3 - 13\alpha^2 + 2\alpha + 6 = 0.$$

Then

(i) for $\alpha_0 \le \alpha \le 1$, F is starlike in

$$|z| < r_1 \equiv (2-\sqrt{2})/\alpha ,$$

(ii) for $0 < \alpha \le \alpha_0$, F is starlike in

$$|z| < r_2 \equiv \left[\frac{4(1-\alpha)}{(1-\alpha)(2-\alpha)+ \sqrt{(1-\alpha)(4+8\alpha-3\alpha^2-\alpha^3)}} \right]^{1/2} .$$

The bounds for $|z|$ in (i) and (ii) are sharp.

This result is obtained by replacing α by $1-\alpha$ and β by $1/2$ in Theorem 3.4.1.

Also, for $\beta = 1/2$ in Theorem 3.4.1, we deduce the following:

Corollary 3.4.1c: Let $F \in \tilde{V}^*(1-\alpha)$ and let $\alpha_0 (= 0.145\ldots$approx) be the smallest positive root of the equation

$$\alpha^4 - 6\alpha^3 - \alpha^2 + 14\alpha - 2 = 0.$$

Then

(i) for $0 \le \alpha \le \alpha_0$, F is starlike in

$$|z| < r_1 \equiv (2-\sqrt{2})/2(1-\alpha)$$

(ii) for $\alpha_0 \le \alpha < 1$, F is starlike in

$$|z| < r_2 \equiv \left[\frac{4\alpha}{\alpha(1+\alpha)+ \sqrt{\alpha(8+\alpha-6\alpha^2+\alpha^3)}} \right]^{1/2} .$$

The bounds for $|z|$ in (i) and (ii) are sharp.

Putting $(\alpha,\beta) = (0, (2\delta-1)/2\delta)$ and $(\alpha,\beta) = (0,1/2)$ in Theorem 3.4.1, we deduce the following corresponding results for functions in $\bar{V}(\delta)$ and $\tilde{V}^*$.

Corollary 3.4.1d: Each function $F \in \bar{V}(\delta)$ maps $|z| < [1+\sqrt{\frac{(2\delta-1)}{2\delta}}]^{-1}$ onto a starlike region. The result is sharp.

Corollary 3.4.1e: Each function $F \in \tilde{V}^*$ maps $|z| < (2-\sqrt{2})$ onto a starlike region. The result is sharp.

Remark 3.4.1: (a) Putting $\beta = 1$ in Theorem 3.4.1, we get the corresponding result for functions in V_α^* obtained by Singh and Goel [78].

(b) $(\alpha,\beta) = (0,1)$ in Theorem 3.4.1 leads to the corresponding result due to Livingston [39].

(c) Subtracting γ from both sides of equation (3.4.4) and proceeding on the similar lines as in Theorem 3.3.1, we get radii of starlikeness of order $\gamma(0 \leq \gamma < 1)$ for functions of the form (3.1.1) where $f \in S^*(\alpha,\beta)$. The results so obtained will generalize the corresponding results obtained separately by Libera and Livingston [37], Bajpai and Singh [4] and Al-Amiri [2].

CHAPTER IV

ON A CLASS OF UNIVALENT FUNCTIONS WHOSE DERIVATIVES HAVE A POSITIVE REAL PART

4.1 Let f be analytic in a convex domain E. If f satisfies the condition

$$(4.1.1) \qquad \mathrm{Re}\,(f'(z)) > 0$$

for all $z \in \Delta$, then it is well known (see [57], [82] and others) that f is univalent in E. MacGregor [41] investigated the properties, e.g., coefficient estimates, distortion theorems, radius of convexity etc. for functions f which are analytic in Δ, have power series representation

$$(4.1.2) \qquad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and satisfy (4.1.1) for $z \in \Delta$. We denote the class of such functions by R. Analogous properties have also been obtained for analytic functions with initial zero coefficients in (4.1.2) and satisfying (4.1.1) for $z \in \Delta$. Ezrohi [14] and Martynov [49] obtained the radius of convexity along with the other properties for the class R_α of functions that are analytic and satisfy

$$(4.1.3) \qquad \mathrm{Re}\,(f'(z)) > \alpha$$

for $0 \leq \alpha < 1$, $z \in \Delta$. Several other subclasses of analytic functions whose derivatives have positive real part in the unit disc have also

been studied by MacGregor [44, 46], Goel [18, 19], Padmanabhan [64], Caplinger and Causey [10], Shaffer [75] and others.

On lines similar to those adopted in Chapter II, we now propose a unified approach to the study of various subclasses of univalent functions whose derivatives have a positive real part in the unit disc. Thus, we introduce the class $R(\alpha,\beta)$ which, for different values of the parameters α, $\beta (0 \leq \alpha < 1,\ 0 < \beta \leq 1)$, not only gives rise to the classes studied by the above mentioned workers but also gives rise to many new subclasses of univalent functions. Thus we have the following definition:

Definition 4.1.1: Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc Δ. Then $f \in R(\alpha,\beta)$ if

$$(4.1.4) \qquad \left|(f'(z)-1)/\{2\beta(f'(z)-\alpha)-(f'(z)-1)\}\right| < 1$$

holds for some α, β $(0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ and all $z \in \Delta$

It is easy to check that $R(\alpha,1)$ is the class R_α studied by Ezrohi [14], Martynov [49] etc.; $R(0,1) \equiv R$ and $R(0,1/2) \equiv \bar{R}$ give the classes introduced and studied by MacGregor [41, 46] while the cases $(\alpha,\beta) = (0,(2\delta-1)/2\delta), \delta > 1/2$, $(\alpha,\beta) = ((1-\gamma)/(1+\gamma),\ (1+\gamma)/2)$ and $(\alpha,\beta) = (0,1-\delta)$ in (4.1.4) lead respectively to the classes $\bar{R}(\delta)$, $R(\gamma)$ and $\bar{R}^{**}(\delta)$ studied earlier by Goel [18], Padmanabhan [64], Caplinger and Causey [10], Shaffer [75] etc.; also a replacement of α by $1-\alpha$ and β by $1/2$ in (4.1.4) gives the class $\bar{R}^{*}(\alpha)$ introduced by Goel [20].

From the definition given above it is clear that $R(\alpha,\beta)$ is a subclass of the class of functions whose derivatives have a positive real part in Δ and hence a function in $R(\alpha,\beta)$ is univalent in Δ. It is easily seen that for $f \in R(\alpha,\beta)$, the values $f'(z)$ lie inside the circle in the right half-plane with centre at $(1+\alpha-2\alpha\beta)/2(1-\beta)$ and radius $(1-\alpha)/2(1-\beta)$. Further, it follows from Schwarz's Lemma that if $f \in R(\alpha,\beta)$, then $f'(z) = (1+(2\alpha\beta-1)z\,\phi(z))/(1+(2\beta-1)z\phi(z))$ for some $\phi \in A$.

In the present chapter, we determine sharp coefficient estimates, distortion theorems, radii of convexity etc., for the class $R_k(\alpha,\beta)$ consisting of functions whose power series begins $f(z) = z + a_{k+1}z^{k+1} + a_{k+2}z^{k+2}+\ldots$ and which satisfy (4.1.4) for some $\alpha,\beta(0 \leq \alpha < 1, 0 < \beta \leq 1)$ and $z \in \Delta$. A sufficient condition for a function to be in $R_k(\alpha,\beta)$ has also been obtained. For different values of the parameters $\alpha,\beta(0 \leq \alpha < 1, 0 < \beta \leq 1)$ our results sharpen and generalize the corresponding results obtained by MacGregor [41, 46], Goel [18, 19], Ezrohi [14], Martynov [49], Padmanabhan [64], Caplinger and Causey [10] and Shaffer [75] etc.

Remark 4.1.1: The function $f(z)$, given by (4.1.2) and satisfying

(4.1.5) $\quad |(f'(z)-1)/(f'(z)+1-2\alpha)| < \beta$

for some α,β $(0 \leq \alpha < 1, 0 < \beta \leq 1)$, $z \in \Delta$ is easily obtained by replacing α by $(1-\beta+2\alpha\beta)/(1+\beta)$ and β by $(1+\beta)/2$ in (4.1.4). The class of functions satisfying (4.1.5) was introduced and studied in [29].

4.2 Coefficient estimates:

Theorem 4.2.1: *If* $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ *is in* $R_k(\alpha,\beta)$, *then* $|a_n| \leq 2\beta(1-\alpha)/n$ *for* $n \geq k+1$, $k = 1,2,\ldots$. *The inequality is sharp for each* n.

Proof: Since $f \in R_k(\alpha,\beta)$, by Schwarz's Lemma, we have

$$f'(z) = \frac{1+(2\alpha\beta-1)\,\omega(z)}{1+(2\beta-1)\,\omega(z)} \tag{4.2.1}$$

where $\omega(z) = \sum_{m=k}^{\infty} t_m z^m = z^k \phi(z)$ is in $\mathcal{B}$. Then (4.2.1) gives

$$((2\beta-1)\,f'(z)-(2\alpha\beta-1))\,\omega(z) = 1-f'(z)$$

or

$$[2\beta(1-\alpha) + \sum_{m=k+1}^{\infty} (2\beta-1)m\, a_m z^{m-1}]\,[\sum_{m=k}^{\infty} t_m z^m] = -\sum_{m=k+1}^{\infty} m\, a_m z^{m-1}. \tag{4.2.2}$$

Equating corresponding coefficients on both sides of (4.2.2) we observe that the coefficient a_n on the right of (4.2.2) depends only on $a_{k+1}, a_{k+2}, \ldots, a_{n-1}$ on the left of (4.2.2). Hence for $n \geq k+1$, it follows from (4.2.2) that

$$[2\beta(1-\alpha) + \sum_{m=k+1}^{n-k} (2\beta-1)m\, a_m z^{m-1}]\,\omega(z) = -\sum_{m=k+1}^{n} m a_m z^{m-1} - \sum_{m=n+1}^{\infty} b_m z^{m-1}$$

say. Since $|\omega(z)| < 1$, we get

$$|2\beta(1-\alpha) + \sum_{m=k+1}^{n-k} (2\beta-1)m a_m z^{m-1}| \geq |\sum_{m=k+1}^{n} m a_m z^{m-1} + \sum_{m=n+1}^{\infty} b_m z^{m-1}|. \tag{4.2.3}$$

Squaring both sides of (4.2.3) and integrating round $|z| = r$, $0 < r < 1$, we obtain

$$4\beta^2(1-\alpha)^2 + \sum_{m=k+1}^{n-k} (2\beta-1)^2 m^2 |a_m|^2 r^{2m-2} \geq \sum_{m=k+1}^{n} m^2 |a_m|^2 r^{2m-2} + \sum_{m=n+1}^{\infty} |b_m|^2 r^{2m-2} .$$

If we take the limit as r approaches 1, then

$$4\beta^2(1-\alpha)^2 + \sum_{m=k+1}^{n-k} (2\beta-1)^2 m^2 |a_m|^2 \geq \sum_{m=k+1}^{n} m^2 |a_m|^2$$

or

$$4\beta(1-\beta) \times \sum_{m=k+1}^{n-k} m^2 |a_m|^2 + n^2 |a_n|^2 \leq 4\beta^2(1-\alpha)^2.$$

Since $0 < \beta \leq 1$, this gives,

$$n^2 |a_n|^2 \leq 4\beta^2(1-\alpha)^2$$

whence follows that

$$|a_n| \leq \frac{2\beta(1-\alpha)}{n}, \quad n \geq k+1 .$$

The bounds are sharp for the functions

$$f_n(z) = \int_0^z \frac{1+(1-2\alpha\beta)t^{n-1}}{1-(2\beta-1)t^{n-1}} dt$$

for $n \geq k+1$ and $z \in \Delta$.

<u>Remark 4.2.1</u>: Putting k = 1 and different values of the parameters $\alpha, \beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$ in Theorem 4.2.1, we get the corresponding coefficient estimates for the functions in their respective classes

obtained by MacGregor [41, 46], Goel [18] , Padmanabhan [64] , Caplinger and Causey [10] etc.

4.3 Distortion Theorems:

Theorem 4.3.1: *Let* $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ *be in* $\mathcal{R}_k(\alpha,\beta)$. *Then, for* $z \in \Delta$,

$$|f(z)| \leq \int_0^{|z|} \frac{1+(1-2\alpha\beta)t^k}{1-(2\beta-1)t^k}\, dt \tag{4.3.1}$$

and

$$|f(z)| \geq \int_0^{|z|} \frac{1+(2\alpha\beta-1)t^k}{1+(2\beta-1)t^k}\, dt \ . \tag{4.3.2}$$

For $\beta = 1/2$, *the above estimates reduce to*

$$|f(z)| \leq |z| + (1-\alpha)\frac{|z|^{k+1}}{k+1} \tag{4.3.3}$$

and

$$|f(z)| \geq |z| - (1-\alpha)\frac{|z|^{k+1}}{k+1} \ . \tag{4.3.4}$$

All the above estimates are sharp.

Proof: Since $f \in \mathcal{R}_k(\alpha,\beta)$, we observe that condition (4.1.4) coupled with an application of Schwarz's Lemma implies that for $z \in \Delta$, $f'(z)$ assumes values lying in the disk K, on the line segment joining the points $(1+(2\alpha\beta-1)|z|^k)/(1+(2\beta-1)|z|^k)$ and $(1-(2\alpha\beta-1)|z|^k)/(1-(2\beta-1)|z|^k)$ as diameter. Thus, if $g(z) = (1-(2\alpha\beta-1)z^k)/(1-(2\beta-1)z^k)$ then, since $g(0) = f'(0) = 1$ and g is univalent in Δ , it follows that f' is subordinate to g. Hence, we have

$$(4.3.5) \qquad |f'(z)| \le \frac{1-(2\alpha\beta-1)|z|^k}{1-(2\beta-1)|z|^k}$$

and

$$(4.3.6) \qquad \frac{1+(2\alpha\beta-1)|z|^k}{1+(2\beta-1)|z|^k} \le \mathrm{Re}\ \{f'(z)\} \le \frac{1-(2\alpha\beta-1)|z|^k}{1-(2\beta-1)|z|^k} .$$

Further, we have

$$(4.3.7) \qquad f(z) = \int_0^z f'(s)\, ds = \int_0^{|z|} f'(te^{i\theta})\, e^{i\theta}\, dt.$$

Using (4.3.5) in (4.3.7) leads to

$$|f(z)| \le \int_0^{|z|} |f'(te^{i\theta})|dt \le \int_0^{|z|} \frac{1+(1-2\alpha\beta)t^k}{1-(2\beta-1)t^k}\, dt$$

which gives (4.3.1) and (4.3.3).

To prove (4.3.2) and (4.3.4), we note that, (4.3.7) and (4.3.6) give

$$|f(z)| \ge \int_0^{|z|} \mathrm{Re}\ f'(te^{i\theta})dt \ge \int_0^{|z|} \frac{1+(2\alpha\beta-1)t^k}{1+(2\beta-1)t^k}\, dt,$$

which yields (4.3.2) and (4.3.4).

The function $f(z)$ given by

$$f'(z) = \frac{1-(2\alpha\beta-1)z^k}{1-(2\beta-1)\ z^k}$$

shows that all the estimates in (4.3.1) to (4.3.4) are sharp.

For $k = 1$, we deduce the following result:

Corollary 4.3.1a: Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in $R(\alpha,\beta)$, then for $z \in \Delta$

$$(4.3.8) \qquad |f(z)| \leq \int_0^{|z|} \frac{1+(1-2\alpha\beta)t}{1-(2\beta-1)t}\,dt$$

and

$$(4.3.9) \qquad |f(z)| \geq \int_0^{|z|} \frac{1+(2\alpha\beta-1)t}{1+(2\beta-1)t}\,dt$$

where equality holds for the functions given in the theorem with $k = 1$.

Corollary 4.3.1b: Let $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ be in $R_{\alpha,k}$, then for $z \in \Delta$

$$(4.3.10) \qquad |f(z)| \leq \int_0^{|z|} \frac{1-(2\alpha-1)t^k}{1-t^k}\,dt$$

$$(4.3.11) \qquad |f(z)| \geq \int_0^{|z|} \frac{1-(1-2\alpha)t^k}{1+t^k}\,dt.$$

The bounds are sharp.

This result is obtained by putting $\beta = 1$ in Theorem 4.3.1.

Replacing α by $(1-\gamma)/(1+\gamma)$ and β by $(1+\gamma)/2$ in Theorem 4.3.1, we get the following distortion theorems for functions in $R_k(\gamma)$.

Corollary 4.3.1c: Let $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ be in $R_k(\gamma)$, then for $z \in \Delta$

$$(4.3.12) \qquad |f(z)| \leq \int_0^{|z|} \frac{1+\gamma t^k}{1-\gamma t^k}\, dt$$

$$(4.3.13) \qquad |f(z)| \geq \int_0^{|z|} \frac{1-\gamma t^k}{1+\gamma t^k}\, dt$$

The bounds are sharp.

Remark 4.3.1: (a) Putting $(\alpha,\beta) = (0,1/2)$, $(\alpha,\beta) = (0,(2\delta-1)/2\delta), \delta > 1/2$ and replacement of α by $1-\alpha$ and β by $1/2$ in Theorem 4.3.1, we get respectively the corresponding distortion theorems for the functions in the classes $\bar{R}_k$, $\bar{R}_k(\delta)$ and $\bar{R}_k^*(\alpha)$.

(b) $(\alpha,\beta) = (0,1-\delta)$ in Theorem 4.3.1, we obtain distortion theorems for functions in $\bar{R}_k^{**}(\delta)$ determined by Shaffer [75].

(c) $(\alpha,\beta) = (0,1)$ in Theorem 4.3.1, we get the corresponding distortion theorems obtained by MacGregor [41].

(d) For different values of the parameters $\alpha,\beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$ in Corollary 4.3.1a, we get the corresponding distortion theorems for their respective classes obtained by MacGregor [41,46], Goel [18], Padmanabhan [64], Caplinger and Causey [10], Shaffer [75] and others.

4.4 A sufficient condition for a function to be in $R_k(\alpha,\beta)$.

Theorem 4.4.1: Let $f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$ be analytic in Δ. If for some $\alpha, \beta (0 \leq \alpha < 1, 0 < \beta \leq 1/2)$,

$$(4.4.1) \qquad \sum_{n=k+1}^{\infty} (1-\beta) n |a_n| \leq (1-\alpha)\beta ,$$

then $f(z)$ *belongs to* $R_k(\alpha,\beta)$.

Proof: Suppose that (4.4.1) holds for some α,β $(0 \leq \alpha < 1, 0 < \beta \leq 1/2)$ and that

$$f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n ;$$

then for $z \in \Delta$,

$$|f'(z)-1| - |2\beta(f'(z)-\alpha)-(f'(z)-1)|$$

$$= |\sum_{n=k+1}^{\infty} n\, a_n z^{n-1}| - |2\beta(1-\alpha) - \sum_{n=k+1}^{\infty} (1-2\beta)n\, a_n z^{n-1}|$$

$$\leq \sum_{n=k+1}^{\infty} n|a_n|r^{n-1} - \{2\beta(1-\alpha) - \sum_{n=k+1}^{\infty} (1-2\beta)n|a_n|r^{n-1}\}$$

$$< \sum_{n=k+1}^{\infty} n|a_n| - 2\beta(1-\alpha) + \sum_{n=k+1}^{\infty} (1-2\beta)n|a_n|$$

$$= 2\,[\sum_{n=k+1}^{\infty} (1-\beta)n|a_n| - \beta(1-\alpha)]$$

$$\leq 0, \text{ by } (4.4.1).$$

Hence it follows that

$$|(f'(z)-1)/\{2\beta(f'(z)-\alpha)-(f'(z)-1)\}| < 1,$$

so that $f \in R_k(\alpha,\beta)$. Hence the Theorem.

For $k = 1$, we deduce the following result:

Corollary 4.4.1: *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in* Δ . *If for some* α, β $(0 \leq \alpha < 1, 0 < \beta \leq 1/2)$,

$$\sum_{n=2}^{\infty} (1-\beta)n|a_n| \leq (1-\alpha)\beta \quad ,$$

then $f(z)$ belongs to $R(\alpha,\beta)$.

4.5 The radii of convexity for functions in the class $R(\alpha,\beta)$.

Theorem 4.5.1: Let $f \in R(\alpha,\beta)$. For a given $\beta(0 < \beta \leq 1)$ let

$\alpha_o(\beta) = \{-(1+10\beta) + \sqrt{(1+12\beta+36\beta^2+32\beta^3)}\} / 4\beta(4\beta^2-8\beta-1)$. Further,

let $V_1 = \{(\alpha,\beta): 0 \leq \alpha < 1,\ 0 < \beta \leq 1\}$, $\Gamma_1 = \{(\alpha,\beta): 0 \leq \alpha < 1/10,$
$0 < \beta \leq 1\}$, $\Gamma_2 = \{(\alpha,\beta): 1/10 \leq \alpha \leq \alpha_o(\beta),\ 0 < \beta \leq 1\}$,
$\Gamma_3 = V_1 - (\Gamma_1 \cup \Gamma_2)$. Then

(i) f is convex in $|z| < r_1$, for $(\alpha,\beta) \in \Gamma_1 \cup \Gamma_2$ and

(ii) f is convex in $|z| < r_2$, for $(\alpha,\beta) \in \Gamma_3$ where

$$r_1 = [(1-2\alpha\beta) + \sqrt{2\beta(1-\alpha)(1-2\alpha\beta)}\,]^{-1}$$

and

$$r_2 = \left[\frac{\alpha}{\alpha\beta + \sqrt{\alpha(1-2\alpha\beta+\alpha\beta^2)}}\right]^{1/2} .$$

The bounds for $|z|$ in (i) and (ii) are sharp.

Proof: Since $f \in R(\alpha,\beta)$, we have by Schwarz's Lemma

$$f'(z) = \frac{1+(2\alpha\beta-1)\omega(z)}{1+(2\beta-1)\omega(z)} \tag{4.5.1}$$

where $\omega \in B$. Differentiating (4.5.1) logarithmically, we get

$$(4.5.2) \qquad 1+z\frac{f''(z)}{f'(z)} = 1-2\beta(1-\alpha)\left\{\frac{z\omega'(z)}{(1+(2\beta-1)\omega(z))(1+(2\alpha\beta-1)\omega(z))}\right\}.$$

Applying (3.2.2) with $k = 1$, $s = 2\beta-1$, $t = 2\alpha\beta-1$, we obtain

$$(4.5.3) \qquad \mathrm{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} \geq \frac{1}{2\beta(1-\alpha)}\left[\mathrm{Re}\left\{(2\beta-1)p(z)+\frac{2\alpha\beta-1}{p(z)}\right\} - \frac{r^2|(2\beta-1)p(z)+1-2\alpha\beta|^2-|1-p(z)|^2}{(1-r^2)|p(z)|}\right]+\frac{1-2\alpha\beta}{\beta(1-\alpha)}$$

where $p(z) = (1+(2\alpha\beta-1)\omega(z))/(1+(2\beta-1)\omega(z))$.

An application of Lemma 3.2.3 ~~xx (xxxxxx)~~ with $k = 1$, $q = s = 2\beta-1$ and $t = 2\alpha\beta-1$ to (4.5.3) gives

$$(4.5.4) \quad \mathrm{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} \geq \begin{cases} \dfrac{1}{\beta(1-\alpha)(1-r^2)}\Big[\sqrt{4\alpha\beta^2(1-(2\beta-1)r^2)(1+(1-2\alpha\beta)r^2)} \\ \qquad -(1+(2\beta-1)(1-2\alpha\beta)r^2)+(1-2\alpha\beta)(1-r^2)\Big] \\ \qquad\qquad \text{if } R_0 \geq R_1, \\ \dfrac{1-2(1-2\alpha\beta)r+(2\beta-1)(2\alpha\beta-1)r^2}{(1+(2\beta-1)r)(1+(2\alpha\beta-1)r)} \\ \qquad\qquad \text{if } R_0 \leq R_1 \end{cases}$$

where $R_0^2 = \alpha(1+(1-2\alpha\beta)r^2)/(1-(2\beta-1)r^2)$, $R_1=(1+(2\alpha\beta-1)r)/(1+(2\beta-1)r)$.

By (4.5.4) the bound r' of convexity for the class $R(\alpha,\beta)$ is determined either from the equation

$$(4.5.5) \qquad (2\beta-1)(2\alpha\beta-1)r^2-2(1-2\alpha\beta)r+1 = 0$$

or from the equation

$$(4.5.6)\qquad \sqrt{4\alpha\beta^2(1-(2\beta-1)r^2)(1+(1-2\alpha\beta)r^2)}-(1+(1-2\alpha\beta)(2\beta-1)r^2)$$
$$+(1-2\alpha\beta)\,(1-r^2) = 0.$$

Equation (4.5.6) may be reduced to

$$(4.5.7)\qquad (1-2\alpha\beta)r^4 + 2\alpha\beta r^2 - \alpha = 0.$$

Also the two minima given by (4.5.4) become equal to each other for those $(\alpha,\beta)\ \varepsilon\ V_1$ for which

$$(4.5.8)\qquad R_0 = R_1.$$

From (4.5.5) and (4.5.7), we obtain

$$(4.5.9)\qquad r' = r_1 = [(1-2\alpha\beta) + \sqrt{2\beta(1-\alpha)(1-2\alpha\beta)}]^{-1}$$

and

$$(4.5.10)\qquad r' = r_2 = [\frac{\alpha}{\alpha\beta + \sqrt{\alpha(1-2\alpha\beta+\alpha\beta^2)}}]^{1/2}.$$

To obtain the points $(\alpha,\beta)\ \varepsilon\ V_1$ which determine the transition from formula (4.5.9) to formula (4.5.10) we eliminate r from (4.5.5) and (4.5.8) and get

$$(4.5.11)\qquad Q_1(\alpha,\beta) \equiv 2\beta(4\beta^2-8\beta-1)\ \alpha^2+(1+10\beta)\alpha-1 = 0.$$

For a given $\beta(0 < \beta \leq 1)$, the smallest positive root of the equation (4.5.11), which is quadratic in α is given by

$$\alpha_o(\beta) = \frac{-(1+10\beta) + \sqrt{(1+12\beta+36\beta^2+ 32\beta^3)}}{4\beta(4\beta^2-8\beta-1)} .$$

It is evident that $\alpha_o(0) = 1$ and $\alpha_o(1) = 1/10$.

Now, let Γ denote the arc of the curve $Q_1(\alpha,\beta) = 0$ lying in the region $G = \{(\alpha,\beta)\colon 1/10 \leq \alpha < 1, 0 < \beta \leq 1\} \subset V_1$ i.e. passing through the points $(1/10,1)$ and $(1,0)$. The curve Γ divides the region V_1 into two subregions $H = \Gamma_1 \cup \Gamma_2$ and Γ_3 where $\Gamma_1=\{(\alpha,\beta)\colon 0 \leq \alpha < 1/10, 0 < \beta \leq 1\}$, $\Gamma_2=\{(\alpha,\beta)\colon 1/10 \leq \alpha \leq \alpha_o(\beta), 0 < \beta \leq 1\}$ and $\Gamma_3 = V_1-(\Gamma_1 \cup \Gamma_2)$. The curve Γ also gives transition from formula (4.5.9) to formula (4.5.10). It is obvious that Γ has void intersection with Γ_1 so that in Γ_1 we have to use either formula (4.5.9) or formula (4.5.10). But it is easily seen that it is impossible to use formula (4.5.10) for all the points (α,β) lying in $K = \{(\alpha,\beta)\colon\alpha = 0, 0 < \beta \leq 1\} \subset \Gamma_1$. So, formula (4.5.9) must be used for (α,β) lying in Γ_1. Now we consider the points (α,β) lying in the region $G = \Gamma_2 \cup \Gamma_3$. Since the formula (4.5.9) cannot be used for the points (α,β) lying in $W'=\{(\alpha,\beta)\colon \alpha_o'(\beta) \leq \alpha < 1, 1/2 \leq \beta \leq 1$ where $\alpha_o'(\beta) = 1/2\beta\} \subset \Gamma_3$, it therefore follows that for $(\alpha,\beta) \in \Gamma_3$, we have to use formula (4.5.10) while formula (4.5.9) is to be used for $(\alpha,\beta) \in \Gamma_1 \cup \Gamma_2$. This proves (i) and (ii).

Figure 4 shows the transition curve Γ and the regions Γ_1 , Γ_2 and Γ_3.

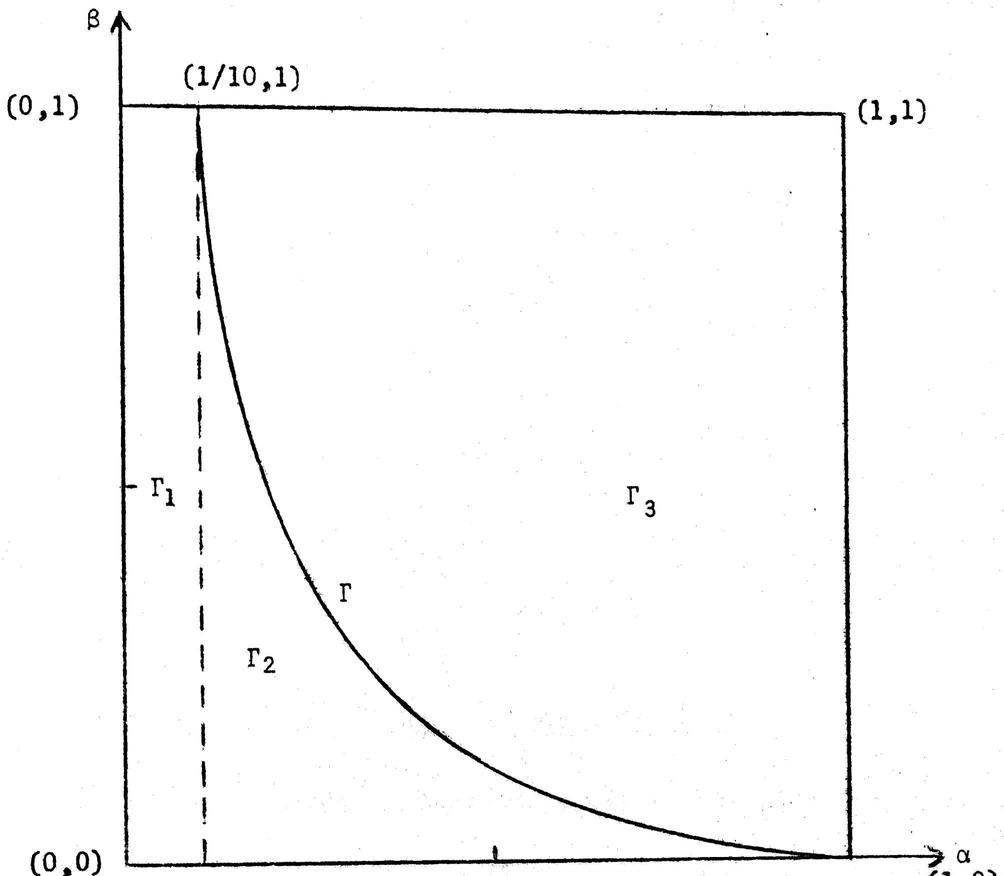

FIG. 4

The functions given by

$$f'(z) = \frac{1-(2\alpha\beta-1)z}{1-(2\beta-1)z}$$

and

$$f'(z) = \frac{1-2\alpha\beta bz + (2\alpha\beta-1)z^2}{1-2\beta bz + (2\beta-1)z^2}$$

where b is determined by the relation

$$\frac{1-2\alpha\beta br + (2\alpha\beta-1)r^2}{1-2\beta br + (2\beta-1)r^2} = \sqrt{\frac{\alpha(1+(1-2\alpha\beta)r^2)}{(1-(2\beta-1)r^2)}} = R_o$$

show that the results obtained in the theorem are sharp.

Corollary 4.5.1a[10, 64] : If $f \in R(\gamma)$, $0 < \gamma \leq 1$, then

(i) f maps $|z| < (\sqrt{2}-1)/\gamma$ onto a convex domain if

$$\frac{(\sqrt{2}-1)(\sqrt{3}+1)}{\sqrt{2}} \leq \gamma \leq 1,$$

(ii) f maps

$$|z| < [\{(\gamma^2-1) + \sqrt{(1-\gamma^2)(1+4\gamma-\gamma^2)}\}/2\gamma(1+\gamma)]^{1/2}$$

onto a convex domain if

$$0 < \gamma \leq (\sqrt{2}-1)(\sqrt{3}+1)/\sqrt{2}.$$

The bounds for $|z|$ in (i) and (ii) are both sharp.

The above result is obtained by replacing α by $(1-\gamma)/(1+\gamma)$ and β by $(1+\gamma)/2$ in Theorem 4.5.1. It is to be noted that this result was determined by Padmanabhan [64] and Caplinger and Causey [10] separately by using different techniques. Also $\gamma = 1$ in the above corollary gives a result due to MacGregor [41].

Putting $\beta = 1$ in Theorem 4.5.1, we get the following result due to Ezrohi [14] and Martynov [49].

Corollary 4.5.1b: If $f \in R_\alpha$, $0 \leq \alpha < 1$, then

(i) f maps $|z| < [(1-2\alpha) + \sqrt{2(1-\alpha)(1-2\alpha)}]^{-1}$ onto a convex domain if

$$1/10 \leq \alpha < 1,$$

(ii) f <u>maps</u> $|z| < [\alpha/\{\alpha + \sqrt{\alpha(1-\alpha)}\}]^{1/2}$ <u>onto a convex domain if</u>

$$0 \leq \alpha \leq 1/10.$$

<u>The bounds for</u> $|z|$ <u>in</u> (i) <u>and</u> (ii) <u>are sharp</u>.

<u>Corollary 4.5.1c</u>[19, 30]: <u>Let</u> $f \in \bar{R}^*(\alpha)$ <u>and let</u> $\alpha_0 = (1+\sqrt{5})/4$. <u>Then</u>

(i) <u>for</u> $\alpha_0 \leq \alpha \leq 1$, f <u>is convex in</u>

$$|z| < 1/2\alpha \quad ,$$

(ii) <u>for</u> $0 < \alpha \leq \alpha_0$, f <u>is convex in</u>

$$|z| < \left[\frac{(\alpha-1) + \sqrt{(1-\alpha)(1+3\alpha)}}{2\alpha}\right]^{1/2} .$$

<u>The bounds for</u> $|z|$ <u>in</u> (i) <u>and</u> (ii) <u>are both sharp</u>.

This result is obtained by replacing α by $1-\alpha$ and β by $1/2$ in Theorem 4.5.1. It can be noted that this result was determined by Kaczmarski [30] and Goel [19] separately by using different techniques.

The cases $(\alpha,\beta) = (0,(2\delta-1)/2\delta), \delta > 1/2$, $(\alpha,\beta) = (0,1-\delta)$ in Theorem 4.5.1 lead respectively to the following results due to Goel [18] and Shaffer [75] .

<u>Corollary 4.5.1d</u>: <u>Each function</u> f(z) <u>in</u> $\bar{R}(\delta)$ <u>maps</u> $|z| < [1 + \sqrt{(2\delta-1)/\delta}\,]^{-1}$ <u>onto a convex domain</u>. <u>The result is sharp</u>.

<u>Corollary 4.5.1e</u>: <u>Each function</u> f(z) <u>in</u> $\bar{R}^{**}(\delta)$ <u>maps</u> $|z| < [1 + \sqrt{2(1-\delta)}\,]^{-1}$ <u>onto a convex domain</u>. <u>The result is sharp</u>.

Also, putting $(\alpha,\beta) = (0, 1/2)$ in Theorem 4.5.1, we get the following result due to MacGregor [46].

Corollary 4.5.1f: Each function $f \in \bar{R}$ maps $|z| < 1/2$ onto a convex domain. The result is sharp.

Remark 4.5.1: Subtracting γ from both sides of the equation (4.5.4) and proceeding on similar lines as in Theorem 4.5.1, we may get the radii of convexity of order γ for functions in $R(\alpha,\beta)$.

We now obtain the radii of convexity for functions belonging to the class $R_k(\alpha,\beta)$.

Theorem 4.5.2: Let $f \in R_k(\alpha,\beta)$ and let $r_0'(\alpha,\beta,k)$ be the smallest positive root of the equation

$$k((2\beta-1)(2\alpha\beta-1)r^{2k}-1)(1-r^2) + 2(2\beta-1)(2\alpha\beta-1)r^{2k+1}+2r(1-2(1-\beta-\alpha\beta)r^k)=0.$$

Then

(i) for $0 < r \leq r_0'(\alpha,\beta,k)$, f is convex in

$$|z| < r_1 \equiv [(1-\beta(1+\alpha)+\beta(1-\alpha)k)+\sqrt{\beta^2(1-\alpha)^2(1+k)^2+2\beta(1-\alpha)(1-\beta-\alpha\beta)k}\,]^{-1/k}$$

(ii) for $r_0'(\alpha,\beta,k) \leq r < 1$, f is convex in

$$|z| < r_2 ,$$

where r_2 is the smallest positive root of the equation

$$2(2\beta-1)(1-2\alpha\beta)r^{2k}(1-r^2)-(\beta(1-\alpha)\ (1+k^2)+2(1-\beta-\alpha\beta)k)\ r^{k-1}\ (1+r^4)$$

$$+\ (4(1-\beta-\alpha\beta)k-2\beta(1-\alpha)(1-k^2))r^{k+1}+2(1-r^2) = 0.$$

The bounds for $|z|$ in (i) and (ii) are both sharp.

Proof: Since $f \in R_k(\alpha,\beta)$, we have by Lemma 2.2.1

$$(4.5.12) \qquad f'(z) = \frac{1+(2\alpha\beta-1)z^k\phi(z)}{1+(2\beta-1)z^k\phi(z)}$$

for some $\phi \in A$ and all $z \in \Delta$. Writing $z\phi(z) = \omega(z)$ where $\omega \in B$, we get

$$(4.5.13) \qquad f'(z) = \frac{1+(2\alpha\beta-1)z^{k-1}\omega(z)}{1+(2\beta-1)z^{k-1}\omega(z)}.$$

Differentiating (4.5.13) logarithmically, we have

$$(4.5.14) \quad 1+z\frac{f''(z)}{f'(z)} = 1-2\beta(1-\alpha)\{\frac{z^k\omega'(z)+(k-1)z^{k-1}\omega(z)}{(1+(2\beta-1)z^{k-1}\omega(z))(1+(2\alpha\beta-1)z^{k-1}\omega(z))}\}.$$

Applying (3.2.2) with $s = 2\beta-1$, $t = 2\alpha\beta-1$ to (4.5.14), we obtain

$$(4.5.15) \quad \text{Re}\ \{1+z\frac{f''(z)}{f'(z)}\} \geq \frac{1}{2\beta(1-\alpha)}\ [\text{Re}\ \{(2\beta-1)\ kp(z)+\frac{(2\alpha\beta-1)k}{p(z)}\}$$

$$-\frac{r^{2k}|(2\beta-1)p(z)+1-2\alpha\beta|^2-|1-p(z)|^2}{r^{k-1}(1-r^2)\,|p(z)|}]$$

$$+\ 1+\frac{(1-\beta-\alpha\beta)k}{\beta(1-\alpha)}$$

where $p(z) = (1+(2\alpha\beta-1)z^{k-1}\omega(z))/(1+(2\beta-1)z^{k-1}\omega(z))$.

An application of Lemma 3.2.3 with $q = (2\beta-1)k$, $s = 2\beta-1$, $t = 2\alpha\beta-1$ to (4.5.15) gives

$$(4.5.16)\quad \mathrm{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} \geq \begin{cases} \dfrac{1}{\beta(1-\alpha)r^{k-1}(1-r^2)}\left[\sqrt{N}-(1-(2\beta-1)(2\alpha\beta-1)r^{2k}) + \right. \\ \qquad \left. + ((1-\beta)k+\beta(1-\alpha-\alpha k))r^{k-1}(1-r^2)\right] \\ \qquad\qquad \text{if } R_o \geq R_1, \\ \dfrac{1-(1-\beta(1+\alpha)+\beta k(1-\alpha))r^k+(2\beta-1)(1-2\alpha\beta)r^{2k}}{(1+(2\beta-1)r^k)\,(1+(2\alpha\beta-1)r^k)} \\ \qquad\qquad \text{if } R_o \leq R_1 \end{cases}$$

where

$$N = [\{(2\beta-1)kr^{k-1}(1-r^2)+1-(2\beta-1)^2r^{2k}\}\{1+(2\alpha\beta-1)kr^{k-1}(1-r^2) - (2\alpha\beta-1)^2\, r^{2k}\}]^{1/2},$$

$$R_o^2 = \frac{1+k(2\alpha\beta-1)r^{k-1}(1-r^2)-(1-2\alpha\beta)^2\, r^{2k}}{1+(2\beta-1)kr^{k-1}(1-r^2)-(2\beta-1)^2r^{2k}} \quad \text{and} \quad R_1 = \frac{1+(2\alpha\beta-1)r^k}{1+(2\beta-1)r^k}.$$

Now proceeding on the similar lines as in Theorem 3.3.2, (4.5.16) gives the required results easily.

The functions given by

$$f'(z) = \frac{1-(2\alpha\beta-1)z^k}{1-(2\beta-1)z^k}$$

and

$$f'(z) = \frac{1-(1+(2\alpha\beta-1)z^{k-1})bz + (2\alpha\beta-1)z^{k+1}}{1-(1+(2\beta-1)\, z^{k-1})bz + (2\beta-1)z^{k+1}}$$

where b is determined by the relation

$$\frac{1-(1+(2\alpha\beta-1)r^{k-1})br+(2\alpha\beta-1)r^{k+1}}{1-(1+(2\beta-1)r^{k-1})br+(2\beta-1)r^{k+1}} = \sqrt{\frac{1+k(2\alpha\beta-1)r^{k-1}(1-r^2)-(1-2\alpha\beta)^2 r^{2k}}{1+(2\beta-1)kr^{k-1}(1-r^2)-(2\beta-1)^2 r^{2k}}} = R_o$$

show that the results obtained in the theorem are sharp.

For $\beta = 1$, we deduce the following result:

Corollary 4.5.2a: *Let $f \in R_{\alpha,k}$ and let $r_o'(\alpha, k)$ be the smallest positive root of the equation*

$$k((2\alpha-1)\, r^{2k}-1)\,(1-r^2) + 2(2\alpha-1)\, r^{2k+1} + 2r(1+2\alpha r^k) = 0.$$

Then

(i) *for* $0 < r \leq r_o'(\alpha,k)$, f *is convex in*

$$|z| < r_1 \equiv [(k-\alpha-\alpha k) + \sqrt{(1-\alpha)\,\{(1-\alpha)(1+k^2)-2\alpha k\}}]^{-1/k},$$

(ii) *for* $r_o'(\alpha, k) \leq r < 1$, f *is convex in*

$$|z| < r_2,$$

where r_2 *is the smallest positive root of the equation*

$$2(1-2\alpha)r^{2k}(1-r^2)-((1-\alpha)(1+k^2)-2\alpha k)r^{k-1}(1+r^4)-(2(1-\alpha)(1-k^2)+4\alpha k)r^{k+1} + 2(1-r^2) = 0.$$

The bounds for $|z|$ in (i) and (ii) are sharp.

Corollary 4.5.2b: *Let $f \in R_k(\gamma)$ and let $\bar{r}(\gamma, k)$ be the smallest positive root of the equation*

$$k(1+\gamma^2 r^{2k})(1-r^2)-2r(1-\gamma^2 r^{2k}) = 0.$$

<u>Then</u>

(i) <u>for</u> $0 < r \leq \bar{r}(\gamma,k)$, f <u>is convex in</u>

$$|z| < r_1 \equiv [\gamma(k+\sqrt{(k^2+1)})]^{-1/k},$$

(ii) <u>for</u> $\bar{r}(\gamma,k) \leq r < 1$, f <u>is convex in</u>

$$|z| < r_2,$$

<u>where</u> r_2 <u>is the smallest positive root of the equation</u>

$$2\gamma^2 r^{2k}(1-r^2)-\gamma(1+k^2)r^{k-1}(1+r^4)-2\gamma(1-k^2)r^{k+1}-2r^2+2 = 0.$$

<u>The bounds for</u> $|z|$ <u>in</u> (i) <u>and</u> (ii) <u>are sharp</u>.

The above result is obtained by replacing α by $(1-\gamma)/(1+\gamma)$ and β by $(1+\gamma)/2$ in Theorem 4.5.2.

Also for the case $(\alpha,\beta) = (0,(2\delta-1)/2\delta)$, $\delta > 1/2$ or replacement of α by $1-\alpha$ and β by $1/2$ in Theorem 4.5.2, we deduce the following results:

<u>Corollary 4.5.2c</u>: <u>Let</u> $f \in \bar{R}_k(\delta)$ <u>and let</u> $r_k(\delta)$ <u>be the smallest positive root of the equation</u>

$$k((1-\delta)r^{2k}-\delta)(1-r^2) + 2(1-\delta)\, r^{2k+1} + 2\delta r(\delta-r^k) = 0.$$

<u>Then</u>

(i) <u>for</u> $0 < r \leq r_k(\delta)$, f <u>is convex in</u>

$$|z| < r_1 \equiv [\frac{1}{2\delta}\{(1+(2\delta-1)k)+\sqrt{(2\delta-1)^2(1+k^2)+2k(2\delta-1)}\}]^{-1/k},$$

(ii) for $r_k(\delta) \leq r < 1$, f is convex in

$$|z| < r_2,$$

where r_2 is the smallest positive root of the equation

$$4(1-\delta)r^{2k}(1-r^2)+(2k+(2\delta-1)(1+k^2))r^{k-1}(1+r^4)-2(2k-(2\delta-1)(1-k^2))r^{k+1} - 4\delta(1-r^2) = 0.$$

The bounds for $|z|$ in (i) and (ii) are sharp.

Corollary 4.5.2d: Let $f \in \overline{R}^*_k(\alpha)$ and let $r(\alpha,k)$ be the smallest positive root of the equation

$$2\alpha\, r^{k+1}-kr^2-2r+k = 0.$$

Then

(i) for $0 < r \leq r(\alpha,k)$, f is convex in

$$|z| < r_1 = [\alpha(1+k)]^{-1/k},$$

(ii) for $r(\alpha,k) \leq r < 1$, f is convex in

$$|z| < r_2,$$

where r_2 is the smallest positive root of the equation

$$\alpha(1+k)^2 r^{k-1}(1+r^4) - 2\alpha(k^2+2k-1)\, r^{k+1} + 4r^2-4 = 0.$$

The bounds for $|z|$ in (i) and (ii) are sharp.

Remark 4.5.2: (a) Putting $(\alpha,\beta) = (0,1-\delta)$ in Theorem 4.5.2, we get the expression for the exact radius of convexity for functions in

$R_k^{**}(\delta)$. This result was obtained by Shaffer [75] by using a different technique.

(b) $(\alpha,\beta) = (0,1/2)$ or $(\alpha,\beta) = (0,1)$ in Theorem 4.5.2 leads to the corresponding results determined by MacGregor [41, 46].

CHAPTER V

ON RADII OF STARLIKENESS AND CONVEXITY OF SOME CLASSES OF ANALYTIC FUNCTIONS

5.1 Let $\mathcal{D}^*_k(\alpha,\beta)$ be the class of functions

(5.1.1) $$f(z) = z + \sum_{n=k+1}^{\infty} a_n z^n$$

analytic in Δ and satisfying

(5.1.2) $$\text{Re}\,\{\frac{f(z)}{g(z)}\} > 0, \quad z \in \Delta,$$

for some $g \in S^*_k(\alpha,\beta)$. Denote by $E_k(\alpha,\beta)$, the class of functions f given by (5.1.1) and satisfying

(5.1.3) $$\text{Re}\,\{\frac{f'(z)}{g'(z)}\} > 0, \quad z \in \Delta,$$

for some $g \in C_k(\alpha,\beta)$.

Let $\sum$ denote the class of all non-constant analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in Δ satisfying

(5.1.4) $$\text{Re}\,\{(1-z^2)f'(z)\} \geq 0, \quad z \in \Delta.$$

Let $F(\alpha,\beta,\lambda)$ represent the subclass of S consisting of functions of the form

(5.1.5) $$g_\lambda(z) = \lambda f(z) + (1-\lambda)z$$

where $f \in R(\alpha,\beta)$ and λ is a real number satisfying $0 < \lambda < 1$.

Let $G^*(\alpha,\beta,\lambda)$ denote the class of functions

$$(5.1.6) \qquad F_\lambda(z) = (1-\lambda)\, f(z) + \lambda z f'(z)$$

for some $f \in S^*(\alpha, \beta)$ and $0 < \lambda < 1$. Denote by $\bar{G}^*(\alpha, \beta, \lambda)$, the class of functions F_λ of the form (5.1.6) where $f \in C(\alpha, \beta)$.

Let $H_k^*(\alpha, \delta, \lambda)$ be the class of functions f, given by (5.1.1), and satisfying

$$(5.1.7) \qquad \left| \frac{f(z)}{\lambda f(z) + (1-\lambda) g(z)} - 1 \right| < \delta, \; 0 < \delta \leq 1, \; z \in \Delta ,$$

for some $g \in S^*_{\alpha,k}$ and $0 \leq \lambda < 1$. Denote by $\bar{H}^*_k(\gamma, \delta, \lambda)$, the class of functions f, given by (5.1.1), and satisfying (5.1.7) for some $g \in S^*_k(\gamma)$ and $0 \leq \lambda < 1$.

Represent by $Q_k(\alpha, \delta, \lambda)$, the class of functions f given by (5.1.1) and satisfying

$$(5.1.8) \qquad \left| \frac{f'(z)}{\lambda f'(z) + (1-\lambda) g'(z)} - 1 \right| < \delta, \; 0 < \delta \leq 1, \; z \in \Delta,$$

for some $g \in C_{\alpha,k}$ and $0 \leq \lambda < 1$. Also denote by $\bar{Q}_k(\gamma, \delta, \lambda)$, the class of functions f given by (5.1.1) and satisfying (5.1.8) for some $g \in C_k(\gamma)$ and $0 \leq \lambda < 1$.

In the present chapter, we propose to determine the radii of starlikeness and convexity for functions of the above mentioned classes. The intrinsic strength of the methods used in this chapter lies in Lemma 3.2.3 which is widely applied throughout this chapter. Our results sharpen and generalize the various results obtained by Livingston [39], MacGregor [44, 45], Shah [77], Ratti [70], Nikolaeva and Repnina [56],

Hengartner and Schober [25] and many others. It is to be noted that our technique is different from the techniques used by the above mentioned workers.

5.2 The radii of starlikeness for functions in $\mathcal{D}^*_k(\alpha,\beta)$ and radii of convexity for functions in $\mathcal{E}_k(\alpha,\beta)$.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in Δ. MacGregor [44] posed the problem of finding the radius of univalence for functions $f(z)$ satisfying (5.1.2) where $g(z)$ is in S. He himself solved this problem when $g \in S^*$ or C. Later, Ratti [70], Goel [21] etc. determined the radius of univalence (and starlikeness) by considering $g(z)$ in S^*_α. Recently, Bajpai [3] extended the above result by considering the function $f(z)$ given by (5.1.1) and $g \in S^*_{\alpha,k}$. Causey and Merkes [11] also extended the results of MacGregor [44] in a different direction. In this section, we determine the radii of starlikeness for functions in $\mathcal{D}^*_k(\alpha, \beta)$. It will be seen that our results not only include the results of the above mentioned workers but also yield analogous results for the corresponding subclasses of univalent functions when g is associated with various subclasses of starlike functions considered in Sec. 1.6. We also determine the radii of convexity for functions in $\mathcal{E}_k(\alpha,\beta)$. Thus we have the following :

Theorem 5.2.1 : Let $f(z)$ be in $\mathcal{D}^*_k(\alpha,\beta)$. Then f is univalent and starlike for $|z| < r_0$, where r_0 is the smallest positive root of the equation

(5.2.1) $$1+(2\alpha\beta-1-2k)r^k-(1+2k(2\beta-1))r^{2k}-(2\alpha\beta-1)r^{3k} = 0.$$

The result is sharp.

<u>Proof</u> : Since $f \in \mathcal{D}_k^*(\alpha,\beta)$, we have

$$(5.2.2) \qquad \text{Re}\,\{\frac{f(z)}{g(z)}\} > 0$$

for some $g \in S_k^*(\alpha,\beta)$. Applying Lemma 2.2.1 with $(\alpha,\beta) = (0,1)$ to (5.2.2), we get

$$(5.2.3) \qquad f(z) = \frac{1 - z^{k-1}\ \omega(z)}{1 + z^{k-1}\ \omega(z)}\, g(z)$$

where $\omega \in \mathcal{B}$. Differentiating (5.2.3) logarithmically, we have

$$(5.2.4) \quad z\,\frac{f'(z)}{f(z)} = z\,\frac{g'(z)}{g(z)} - 2\,\{\frac{z^k\ \omega'(z) + (k-1)\ z^{k-1}\ \omega(z)}{(1-z^{k-1}\ \omega(z))(1+z^{k-1}\ \omega(z))}\}\ .$$

Applying Lemma 3.2.2 with $s = 1$, $t = -1$ and using (2.3.5) in (5.2.4), we obtain

$$(5.2.5) \quad \text{Re}\,\{z\,\frac{f'(z)}{f(z)}\} \geq \frac{1 + (2\alpha\beta-1)\ r^k}{1 + (2\beta-1)\ r^k} + \frac{1}{2}\,[\,\text{Re}\ \{kp(z) + \frac{(-1)k}{p(z)}\ \} - \frac{r^{2k}\,|p(z) + 1|^2 - |1-p(z)|^2}{r^{k-1}(1-r^2)\ |p(z)|}\,]$$

where $p(z) = (1-z^{k-1}\ \omega(z))/(1+z^{k-1}\ \omega(z))$.

An application of Lemma 3.2.3 with $q = k$, $s = 1$, $t = -1$ to (5.2.5) gives

$$(5.2.6) \quad \text{Re}\,\{z\,\frac{f'(z)}{f(z)}\ \} \geq \frac{1+(2\alpha\beta-1-2k)r^k-(1+2k(2\beta-1))r^{2k}-(2\alpha\beta-1)r^{3k}}{(1+(2\beta-1)r^k)(1-r^{2k})}\ .$$

Thus the function f is starlike if $\text{Re}\,\{zf'(z)/f(z)\} > 0$, which is satisfied, if

(5.2.7) $\quad 1+(2\alpha\beta-1-2k)r^{k}-(1+2k(2\beta-1))r^{2k}-(2\alpha\beta-1)r^{3k} > 0.$

It is easy to check that (5.2.7) is satisfied for $|z| < r_o$, where r_o is the smallest positive root of the equation (5.2.1).

The functions

(5.2.8) $$f(z) = \frac{1 + z^k}{1 - z^k}\, g(z)$$

where $g(z)$ is given by

(5.2.9) $$z\,\frac{g'(z)}{g(z)} = \frac{1 - (2\alpha\beta-1)\, z^k}{1 - (2\beta-1)\, z^k}$$

show that the result obtained in the theorem is sharp.

Putting $\beta = 1$, in Theorem 5.2.1, we get the following result due to Bajpai [3].

<u>Corollary 5.2.1a</u> : <u>Let</u> $f(z)$ <u>be in</u> $\mathcal{D}_k^*(\alpha,1)$. <u>Then</u> f <u>is univalent and starlike for</u> $|z| < r_o$, <u>where</u>

$$r_o = \begin{cases} \left\{\dfrac{\sqrt{(\alpha^2-2\alpha k+2k+k^2)} - (k+1-\alpha)}{2\alpha-1}\right\} & \text{if } \alpha \neq 1/2, \\ (1/2k+1)^{1/k} & \text{if } \alpha = 1/2. \end{cases}$$

<u>The result is sharp for the functions given by</u> (5.2.8) <u>and</u> (5.2.9) <u>with</u> $\beta = 1$.

Putting k= 1 in the above Corollary, we get

Corollary 5.2.1b [21,70] : Let $f(z)$ be in $\mathcal{D}_1^*(\alpha,1)$. Then f is univalent and starlike for $|z| < r_0$, where

$$r_0 = \begin{cases} \{(2-\alpha) + \sqrt{(\alpha^2-2\alpha+3)}\}^{-1} & \text{if } \alpha \neq 1/2, \\ 1/3 & \text{if } \alpha = 1/2 . \end{cases}$$

The result is sharp for the functions given by (5.2.8) and (5.2.9) with $k = 1$, $\beta = 1$.

$k = 1$, $(\alpha,\beta) = (0,1)$ in Theorem 5.2.1 leads to the following result due to MacGregor [44] .

Corollary 5.2.1c : Suppose that $f(z) = z + a_2z^2 + \ldots$ and $g(z) = z + b_2z^2 + \ldots$ are analytic in Δ and g is univalent and starlike in Δ. If $\text{Re}\,\{f(z)/g(z)\} > 0$ for $z \in \Delta$, then f is univalent and starlike in $|z| < 2-\sqrt{3}$. The result is sharp for the functions given by (5.2.8) and (5.2.9) with $k = 1$, $(\alpha, \beta) = (0,1)$.

Remark 5.2.1 : For different values of the parameters $\alpha,\beta(0 \leq \alpha < 1$, $0 < \beta \leq 1)$ in Theorem 5.2.1, analogous results can be obtained where g belongs respectively to the classes $S_k^*(\gamma)$, $\bar{S}_k(\delta)$, $\bar{S}_k^*(\alpha)$ and $\bar{S}_k^*(1-\alpha)$ introduced in Sec. 1.6.

Making use of the fact that $g \in C_k(\alpha,\beta)$ if, and only if, $zg' \in S_k^*(\alpha,\beta)$, the following result follows easily.

Theorem 5.2.2 : Let $f(z)$ be in $E_k(\alpha,\beta)$. Then f is univalent and convex for $|z| < r_0$, where r_0 is the smallest positive root of the

$$1+(2\alpha\beta-1-2k)r^{k}-(1+2k(2\beta-1))r^{2k}-(2\alpha\beta-1)r^{3k} = 0.$$

The result is sharp for the functions given by

$$f'(z) = \frac{1+z^{k}}{1-z^{k}}\, g'(z) \; ; \; 1 + z\,\frac{g''(z)}{g'(z)} = \frac{1-(2\alpha\beta-1)z^{k}}{1-(2\beta-1)z^{k}} \, .$$

Remark 5.2.2 : The results due to Bajpai [3] and Padmanabhan [63] can be obtained by putting $\beta = 1$ and $k = 1$, $\beta = 1$ respectively in Theorem 5.2.2.

5.3 The radius of convexity for functions in $\sum$.

If $f \in \sum$, it is known that f is univalent and $f(\Delta)$ is a domain convex in v-direction, i.e., the intersection of $f(\Delta)$ with each vertical line is connected (or empty). Recently, Hengartner and Schober [25] obtained the radii of convexity for functions in $\sum$. We give an alternative technique to determine the same. Thus we have the following :

Theorem 5.3.1 : If $f(z)$ be in $\sum$, then f maps $|z| < r$ onto a convex region for $r \leq C = 1/2\,(1+\sqrt{5}) - (1/2\,(1+\sqrt{5}))^{1/2} = 0.346 \ldots$. The constant C is sharp for the function

$$f(z) = \frac{i}{2} \log \frac{(1-iz)^2}{(1-z^2)} \, ,$$

which is in $\sum$.

Proof : Since $f \in \sum$, we have

$$\text{(5.3.1)} \qquad \mathrm{Re}\,\{(1-z^2)\,f'(z)\} \geq 0$$

for $z \in \Delta$. Therefore by Schwarz's Lemma, there exists $\omega \in B$ such that

$$(5.3.2) \qquad f'(z) = \frac{1}{1-z^2}\,\frac{1-\omega(z)}{1+\omega(z)}.$$

Differentiating (5.3.2) logarithmically, we have

$$(5.3.3) \qquad 1 + z\frac{f''(z)}{f'(z)} = \frac{1+z^2}{1-z^2} - 2\left\{\frac{z\omega'(z)}{(1-\omega(z))(1+\omega(z))}\right\}.$$

Applying (5.2.2) with $k = 1$, $s = 1$, $t = -1$ and using the estimate $\mathrm{Re}\,((1+z^2)/(1-z^2)) \geq (1-r^2)/(1+r^2)$ in (5.3.3), we get

$$(5.3.4)\ \mathrm{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} \geq \frac{1-r^2}{1+r^2} + \frac{1}{2}\left[\mathrm{Re}\left\{p(z)+\frac{(-1)}{p(z)}\right\} - \frac{r^2|p(z)+1|^2-|1-p(z)|^2}{(1-r^2)|p(z)|}\right]$$

where $p(z) = (1-\omega(z))/(1+\omega(z))$.

An application of Lemma 3.2.3 with $k = q = s = 1$, $t = -1$ to (5.3.4) gives

$$(5.3.5) \qquad \mathrm{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} \geq \frac{1-2r-2r^2-2r^3+r^4}{(1-r^2)(1+r^2)}.$$

Now, by similar arguments as in Theorem 5.2.1, the require result follows easily.

5.4 The radii of convexity for functions in $F(\alpha,\beta,\lambda)$.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in Δ and for a fixed number λ, $0 < \lambda < 1$, consider the function g_λ given by (5.1.5). Recently, Gupta [23] obtained the radii of convexity for the class

of functions given by (5.1.5) where f satisfies the relation

$$(5.4.1) \qquad |f'(z) - 1| < 1, \; z \in \Delta .$$

We now determine the sharp estimates for the radii of convexity for functions in $F(\alpha,\beta,\lambda)$. Thus we prove the following :

<u>Theorem 5.4.1</u> : <u>Let g_λ be in $F(\alpha,\beta,\lambda)$ and let r_0 be the smallest positive root, which exists in $(0,1)$, of the equation</u>

$$(2\beta-1)(2\beta(1-\lambda)+2\beta\alpha\lambda-1)r^4-2(2\beta-1)(2\beta(1-\lambda)+2\alpha\beta\lambda-1)r^3-(4\beta^2+4\beta-2-2\beta\lambda(1+2\beta)(1-\alpha))r^2-2r+1 = 0.$$

<u>Then</u>

(i) g_λ <u>is convex in</u> $|z| < r_1$, <u>if</u> $0 < r \leq r_0$ <u>and</u>

(ii) g_λ <u>is convex in</u> $|z| < r_2$, <u>if</u> $r_0 \leq r < 1$ <u>where</u>

$$r_1 = [(1-2\beta+2\beta\lambda(1-\alpha)) + \sqrt{2\beta\lambda(1-\alpha)(1-2\beta(1-\lambda)-2\alpha\beta\lambda)}]^{-1}$$

<u>and</u>

$$r_2 = \left[\frac{(1-\lambda+\alpha\lambda)}{\beta(1-\lambda+\alpha\lambda)+\sqrt{(1-\lambda+\alpha\lambda)((1-\beta)^2+\beta\lambda(1-\alpha)(2-\alpha))}}\right]^{1/2} .$$

<u>The bounds for $|z|$ in (i) and (ii) are sharp.</u>

<u>Proof</u> : Since $f \in R(\alpha,\beta)$, we have by Schwarz's Lemma

$$(5.4.2) \qquad f'(z) = \frac{1 + (2\alpha\beta-1)\,\omega(z)}{1 + (2\beta-1)\,\omega(z)}$$

where $\omega \in B$. Differentiating (5.1.5) and using (5.4.2), we get

$$(5.4.3) \qquad g'_\lambda(z) = \frac{1 + (2\beta-1-2\beta\lambda(1-\alpha))\ \omega(z)}{1+(2\beta-1)\ \omega(z)} .$$

Logarithmic differentiation gives

$$(5.4.4) \quad 1+z\frac{g''_\lambda(z)}{g'_\lambda(z)} = 1-2\beta\lambda(1-\alpha)\{\frac{z\omega'(z)}{(1+(2\beta-1)\omega(z))(1+(2\beta-1-2\beta\lambda(1-\alpha))\omega(z))}\}.$$

Applying (3.2.2) with $k = 1$, $s = 2\beta-1$, $t = 2\beta-1-2\beta\lambda(1-\alpha)$ to (5.4.4), we obtain

$$(5.4.5) \quad \mathrm{Re}\ \{1+z\frac{g''_\lambda(z)}{g'_\lambda(z)}\} \geq \frac{2\beta(\lambda-\alpha\lambda-1)}{\beta\lambda(1-\alpha)} + \frac{1}{2\beta\lambda(1-\alpha)}\ [\ \mathrm{Re}\ \{(2\beta-1)\ p(z) + \frac{(2\beta-1-2\beta\lambda(1-\alpha))}{p(z)}\} - \frac{r^2|(2\beta-1)p(z)-(2\beta-1-2\beta\lambda(1-\alpha))|^2-|1-p(z)|}{(1-r^2)\ |p(z)|}$$

where $p(z) = (1+(2\beta-1-2\beta\lambda(1-\alpha))\ \omega(z))/(1+(2\beta-1)\ \omega(z))$.

An application of Lemma 3.2.3 with $k=1$, $q = s = 2\beta-1$, $t = 2\beta-1-2\beta\lambda(1-\alpha)$ to (5.4.5) gives

$$(5.4.6) \quad \mathrm{Re}\{1+z\frac{g''_\lambda(z)}{g'_\lambda(z)}\} \geq \begin{cases} \dfrac{1}{\beta\lambda(1-\alpha)(1-r^2)}[\ \sqrt{N^*}-(1-(2\beta-1)(2\beta-1-2\beta\lambda(1-\alpha))r^2) \\ \qquad -(2\beta-1-2\beta\lambda(1-\alpha))(1-r^2)] \\ \qquad\qquad \text{if } R_0 \geq R_1, \\ \dfrac{1+2(2\beta-1-2\beta\lambda(1-\alpha))r+(2\beta-1)(2\beta-1-2\beta\lambda(1-\alpha))r^2}{(1+(2\beta-1)r)(1-(2\beta-1-2\beta\lambda(1-\alpha))r^2)} \\ \qquad\qquad \text{if } R_0 \leq R_1 \end{cases}$$

where

$$(5.4.6;a) \quad N^* = 4\beta^2(1-\lambda+\alpha\lambda)(1-(2\beta-1)r^2)(1-(2\beta-1-2\beta\lambda(1-\alpha))r^2),$$

$$R_o^2 = \frac{(1-\lambda+\alpha\lambda)(1-(2\beta-1-2\beta\lambda(1-\alpha))r^2)}{1-(2\beta-1)r^2} \text{ and } R_1 = \frac{1+(2\beta-1-2\beta\lambda(1-\alpha))r}{1+(2\beta-1)r}.$$

The two minima given by (5.4.6) become equal to each other for those values of α, β and λ for which

$$(5.4.7) \qquad R_o = R_1.$$

The above equation, on simplification, yields

$$(5.4.8) \quad (2\beta-1)(2\beta(1-\lambda)+2\alpha\beta\lambda-1)r^4-2(2\beta-1)(2\beta(1-\lambda)+2\alpha\beta\lambda-1)r^3$$
$$-(4\beta^2+4\beta-2-2\beta\lambda(1-2\beta)(1-\alpha))r^2-2r+1=0.$$

Let r_o be the smallest positive root, which exists in $(0,1)$, of the equation (5.4.8). Then (5.4.6) is equivalent to

$$(5.4.9) \quad \mathrm{Re}\{1+z\frac{g_\lambda''(z)}{g_\lambda'(z)}\} \geq \begin{cases} \dfrac{1}{\beta\lambda(1-\alpha)(1-r^2)}[\sqrt{N^*}-(1-(2\beta-1)(2\beta-1-2\beta\lambda(1-\alpha))r^2) \\ \qquad -(2\beta-1-2\beta\lambda(1-\alpha))(1-r^2)] \\ \qquad\qquad \text{if } r_o \leq r < 1, \\ \dfrac{1+2(2\beta-1-2\beta\lambda(1-\alpha))r+(2\beta-1)(2\beta-1-2\beta\lambda(1-\alpha))r^2}{(1+(2\beta-1)r)(1+(2\beta-1-2\beta\lambda(1-\alpha))r)} \\ \qquad\qquad \text{if } 0 < r \leq r_o \end{cases}$$

where N^* is given by (5.4.6;a).

Thus we see that the radii of convexity r' for functions in $F(\alpha,\beta,\lambda)$ is given by the smallest positive root of the equation

$$(5.4.10) \quad 1+2(2\beta-1-2\beta\lambda(1-\alpha))r+(2\beta-1)(2\beta-1-2\beta\lambda(1-\alpha))r^2=0 \text{ if } 0 < r \leq r_o,$$

and, by the smallest positive root of the equation

(5.4.11) $(2\beta-1-2\beta\lambda(1-\alpha))r^4-2\beta(1-\lambda+\alpha\lambda)r^2+(1-\lambda+\alpha\lambda)=0$ if $r_o \leq r < 1$.

The smallest positive roots of the equations (5.4.10) and (5.4.11) giving the radii of convexity for functions in $F(\alpha,\beta,\lambda)$ are easily seen to be

(5.4.12) $r'=r_1=[(1-2\beta+2\beta\lambda(1-\alpha))+\sqrt{2\beta\lambda(1-\alpha)(1-2\beta(1-\lambda)-2\alpha\beta\lambda)}]^{-1}$ for $0<r\leq r_o$,

and

(5.4.13) $$r'=r_2=\left[\frac{(1-\lambda+\alpha\lambda)}{\beta(1-\lambda+\alpha\lambda)+\sqrt{(1-\lambda+\alpha\lambda)((1-\beta)^2+\beta\lambda(2-\beta)(1-\alpha))}}\right]^{1/2}$$

for $r_o \leq r < 1$ where r_o is the smallest positive root of the equation (5.4.8).

The functions given by

(5.4.14) $$g'_\lambda(z)=\frac{1-(2\beta-1-2\beta\lambda(1-\alpha))z}{1-(2\beta-1)z}$$

and

(5.4.15) $$g'_\lambda(z)=\frac{1-2\beta(1-\lambda+\alpha\lambda)bz+(2\beta-1-2\beta\lambda(1-\alpha))z^2}{1-2\beta bz+(2\beta-1)z^2}$$

where b is determined by the relation

$$\frac{1-2\beta(1-\lambda+\alpha\lambda)br+(2\beta-1-2\beta\lambda(1-\alpha))r^2}{1-2\beta br+(2\beta-1)r^2}=\sqrt{\frac{(1-\lambda+\alpha\lambda)(1-(2\beta-1-2\beta\lambda(1-\alpha))r^2)}{1-(2\beta-1)r^2}}=R_o$$

show that the results obtained in the theorem are sharp.

Putting $\beta=1$, in Theorem 5.4.1, we deduce the following

Corollary 5.4.1a : Let $g_\lambda(z) = \lambda f(z) + (1-\lambda)z$ where $f \in R_\alpha$ and let r_0 be the smallest positive root, which exists in $(0,1)$ of the equation

$$(1-2\lambda+2\alpha\lambda)r^4-2(1-2\lambda+2\alpha\lambda)r^3-6(1-\lambda+\alpha\lambda)r^2-2r + 1 = 0.$$

Then

(i) g_λ is convex in $|z| < r_1$ if $0 < r \leq r_0$ and

(ii) g_λ is convex in $|z| < r_2$ if $r_0 \leq r < 1$ where

$$r_1 = [\sqrt{2\lambda(1-\alpha)(2\lambda(1-\alpha)-1)} - (1-2\lambda+2\alpha\lambda)]^{-1}$$

and

$$r_2 = \left[\frac{(1-\lambda+\alpha\lambda)}{(1-\lambda+\alpha\lambda) + \sqrt{\lambda(1-\alpha)(1-\lambda+\alpha\lambda)}}\right]^{1/2}.$$

The bounds for $|z|$ in (i) and (ii) are sharp for the functions given by (5.4.14) and (5.4.15) with $\beta = 1$.

Putting $(\alpha,\beta) = (0,1/2)$ in Theorem 5.4.1, we get the following result obtained by Gupta [23] by using a different technique.

Corollary 5.4.1b : Let $g_\lambda(z) = \lambda f(z) + (1-\lambda)z$ where f satisfies (5.4.1) and let $\alpha_0 = (\sqrt{5} + 1)/4$. Then

(i) for $\alpha_0 \leq \alpha < 1$, g_λ is convex in

$$|z| < 1/2\lambda,$$

(ii) for $0 < \alpha \leq \alpha_0$, g_λ is convex in

$$|z| < \left[\frac{\sqrt{(1-\lambda)(1+3\lambda)} - (1-\lambda)}{2\lambda}\right]^{1/2}.$$

The bounds for $|z|$ in (i) and (ii) are sharp for the functions given by (5.4.14) and (5.4.15) with $(\alpha,\beta) = (0,1/2)$.

Remark 5.4.1 : For different values of the parameters α,β $(0 \leq \alpha < 1, 0 < \beta \leq 1)$ in Theorem 5.4.1, analogous results can be obtained, for the classes of functions of the form (5.1.5) where f belongs respectively to the classes defined in Sec. 1.5.

5.5 The radii of starlikeness for functions in $G^*(\alpha,\beta,\lambda)$ and radii of convexity for functions in $\bar{G}^*(\alpha,\beta,\lambda)$.

Livingston [39] proved that if $f \in S^*$ or C then $(1/2)[f(z) + zf'(z)]$ belongs to S^* or C for $|z| < 1/2$. Singh and Goel [78] , Bajpai and Singh [4] etc. extended this result to the case of functions of the class S^*_α. Recently, Nikolaeva and Repnina [56] extended this result to the class of functions given by (5.1.6). Here we first determine exact radii of starlikeness for functions in $G^*(\alpha,\beta,\lambda)$ and deduce the radii of convexity for functions $\bar{G}^*(\alpha,\beta,\lambda)$ by the help of a known fact. Thus we have the following :

Theorem 5.5.1 : Let $F_\lambda(z)$ be in $G^*(\alpha,\beta,\lambda)$ and let r_0 be the smallest positive root, which exists in (0,1) of the equation

$$(5.5.1)\quad (2\beta-2\beta\lambda+2\alpha\beta\lambda-1)(\lambda-1+2\beta(1-2\lambda+\alpha\lambda))r^4+2(1+\lambda(2\beta-1))(2\beta(1-\lambda+\alpha\lambda)-1)r^3+2[\lambda+2\lambda(2\beta-1-\beta\lambda(1-\alpha))+\beta\lambda(1-\alpha)(4\beta-2+\lambda-4\beta\lambda+2\alpha\beta\lambda)]r^2 + 2(1+\lambda-2\beta+2\beta\lambda(1-\alpha))r - (1+\lambda) = 0.$$

Then

(i) F_λ is starlike in $|z| < r_1$, if $0 < r \leq r_0$ and

(ii) F_λ is starlike in $|z| < r_2$, if $r_0 \leq r < 1$ where

$$r_1 = [(1-\beta-\alpha\beta+2\beta\lambda(1-\alpha)) + \sqrt{\beta(1-\alpha)(\beta(1-\alpha)+2\lambda-4\beta\lambda(1-\lambda+\alpha\lambda))}]^{-1}$$

and

$$r_2 = \left[\frac{4\lambda(1-\lambda+\alpha\lambda)-(1-\alpha)}{(4\alpha\beta\lambda(1-\lambda+\alpha\lambda)-(1-\alpha)(1-2\lambda)) + \sqrt{W^*}}\right]^{1/2}$$

where

$$W^* = ((1-\alpha)(1-2\lambda)-4\alpha\beta\lambda(1-\lambda+\alpha\lambda))^2-(4\lambda(1-\lambda+\alpha\lambda)-1+\alpha)(8\alpha\beta\lambda(1-\lambda+\alpha\lambda)-(1-\alpha+4\alpha\lambda)).$$

The bounds for $|z|$ in (i) and (ii) are sharp.

Proof : Since $f \in S^*(\alpha,\beta)$, by Lemma 2.2.1 with $k = 1$, we have

$$z\frac{f'(z)}{f(z)} = \frac{1+(2\alpha\beta-1)\omega(z)}{1+(2\beta-1)\omega(z)} \tag{5.5.2}$$

where $\omega \in \mathcal{B}$. Differentiating (5.1.6) logarithmically and using (5.5.2), we obtain

$$z\frac{F_\lambda'(z)}{F_\lambda(z)} = \frac{1+(2\alpha\beta-1)\omega(z)}{1+(2\beta-1)\omega(z)} - 2\beta\lambda(1-\alpha)\left\{\frac{z\omega'(z)}{(1+(2\beta-1)\omega(z))(1+(2\beta(1-\lambda+\alpha\lambda)-1)\omega(z))}\right\}. \tag{5.5.3}$$

Applying (3.2.2) with $k = 1$, $s = 2\beta-1$ and $t = 2\beta(1-\lambda+\alpha\lambda)-1$ to (5.5.3), we get

$$(5.5.4)\quad \operatorname{Re}\left\{z\frac{F_\lambda'(z)}{F_\lambda(z)}\right\} \geq \frac{1}{2\beta\lambda(1-\alpha)}\left[\operatorname{Re}\left\{(4\beta-1-2\alpha\beta)p(z)+\frac{2\beta(1-\lambda+\alpha\lambda)-1}{p(z)}\right\} - \frac{r^2|(2\beta-1)p(z)-(2\beta(1-\lambda+\alpha\lambda)-1)|^2-|1-p(z)|^2}{(1-r^2)\ |p(z)|}\right] + \left(\frac{1+\alpha\beta-3\beta+2\beta\lambda-2\alpha\beta\lambda}{\beta\lambda(1-\alpha)}\right)$$

where $p(z) = (1+(2\beta(1-\lambda+\alpha\lambda)-1)\omega(z))/(1+(2\beta-1)\omega(z))$.

An application of Lemma 3.2.3 with $k = 1$, $q = 4\beta-1-2\alpha\beta$, $s = 2\beta-1$ and $t = 2\beta(1-\lambda+\alpha\lambda)-1$ to (5.5.4) gives

$$(5.5.5)\ \operatorname{Re}\left\{z\frac{F_\lambda'(z)}{F_\lambda(z)}\right\} \geq \begin{cases} \dfrac{1}{\beta\lambda(1-\alpha)(1-r^2)}\left[\sqrt{M^*}-(1-(2\beta-1)(2\beta-2\beta\lambda+2\beta\alpha\lambda-1)r^2) +(1+\alpha\beta-3\beta+2\beta\lambda-2\alpha\beta\lambda)(1-r^2)\right] & \text{if } R_0 \geq R_1, \\[2ex] \dfrac{1-2(1-\beta-\alpha\beta+2\beta\lambda(1-\alpha))r+(2\alpha\beta-1)(2\beta-1-2\beta\lambda(1-\alpha))r^2}{(1+(2\beta-1)r)(1+(2\beta(1-\lambda+\alpha\lambda)-1)r)} & \text{if } R_0 \leq R_1 \end{cases}$$

where

$$M^* = 4\beta^2(1-\lambda+\alpha\lambda)((2-\alpha)-(2\beta-\alpha)r^2)(1-(2\beta-1-2\beta\lambda(1-\alpha))r^2),$$

$$R_0^2 = \frac{(1-\lambda+\alpha\lambda)(1-(2\beta(1-\lambda+\alpha\lambda)-1)r^2)}{(2-\alpha)-(2\beta-\alpha)r^2} \quad\text{and}\quad R_1 = \frac{1+(2\beta(1-\lambda+\alpha\lambda)-1)r}{1+(2\beta-1)r}.$$

Using similar arguments as in Theorem 5.4.1, (5.5.5) gives the required results easily.

The results obtained in the theorem are sharp for the functions given by

$$(5.5.6) \qquad z\frac{f'(z)}{f(z)} = \frac{1-(2\alpha\beta-1)z}{1-(2\beta-1)z}$$

and

$$(5.5.7) \qquad z\frac{f'(z)}{f(z)} = \frac{1-2\alpha\beta bz+(2\alpha\beta-1)z^2}{1-2\beta bz+(2\beta-1)z^2}$$

where b is determined by the relation

$$(5.5.8) \qquad \frac{1-2\beta(1-\lambda+\alpha\lambda)br+(2\beta(1-\lambda+\alpha\lambda)-1)r^2}{1-2\beta br+(2\beta-1)r^2} = $$
$$= \sqrt{\frac{(1-\lambda+\alpha\lambda)(1-(2\beta(1-\lambda+\alpha\lambda)-1)r^2)}{(2-\alpha)-(2\beta-\alpha)r^2}} = R_o .$$

<u>Corollary 5.5.1</u> : <u>Let</u> $F_\lambda(z)$ <u>be in</u> $G^*(\alpha,1,\lambda)$ <u>and let</u> r_o <u>be the smallest positive root, which exists in</u> $(0,1)$ <u>of the equation</u>

$$(5.5.9) \quad (1-2\lambda+2\alpha\lambda)(1-3\lambda+2\alpha\lambda)r^3+(1-2\lambda+2\alpha\lambda)(1+5\lambda-2\alpha\lambda)r^2-(1-7\lambda+4\alpha\lambda)r-(1+\lambda)=0.$$

<u>Then</u>

(i) F_λ <u>is starlike in</u> $|z| < r_1$, <u>if</u> $0 < r \le r_o$ <u>and</u>

(ii) F_λ <u>is starlike in</u> $|z| < r_2$, <u>if</u> $r_o \le r < 1$ <u>where</u>

$$r_1 = [(-\alpha+2\lambda(1-\alpha)) + \sqrt{(1-\alpha)(1-\alpha-2\lambda+4\lambda^2(1-\alpha))}]^{-1}$$

<u>and</u>

$$r_2 = \left[\frac{4\lambda(1-\lambda+\alpha\lambda)-(1-\alpha)}{(4\alpha\lambda(1-\lambda+\alpha\lambda)-(1-\alpha)(1-2\lambda))+\sqrt{W^{**}}}\right]^{1/2}$$

<u>where</u> W^{**} <u>is given by</u>

$$W^{**}=((1-\alpha)(1-2\lambda)-4\alpha\lambda(1-\lambda+\alpha\lambda))^2-(4\lambda(1-\lambda+\alpha\lambda)-1+\alpha)(8\alpha\lambda(1-\lambda+\alpha\lambda)-(1-\alpha+4\alpha\lambda)).$$

<u>The bounds for</u> $|z|$ <u>in</u> (i) <u>and</u> (ii) <u>are sharp for the functions given by</u> (5.5.6) <u>and</u> (5.5.7) <u>with</u> $\beta = 1$.

The corollary is obtained from the theorem by putting $\beta = 1$. A similar result has been obtained by Nikolaeva and Repnina [56] .

<u>Remark 5.5.1</u> : (a) Putting $\lambda = 1/2$ and $\beta = 1$ in Theorem 5.5.1 and proceeding on the similar lines as in Theorem 3.4.1, we get the result determined by Singh and Goel [78] .

(b) Taking $\lambda = 1/2$ and $(\alpha, \beta) = (0,1)$ in Theorem 5.5.1, we obtain the corresponding result due to Livingston [39] .

(c) For different values of the parameters α, β $(0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ in Theorem 5.5.1, our results yield, analogous results for the classes of functions of the form (5.1.6) where f belong to the classes $\bar{S}(\delta)$, $S^*(\gamma)$, $\bar{S}^*(1-\alpha)$ and $\bar{S}^*(\alpha)$ respectively.

<u>Theorem 5.5.2</u> : <u>Let</u> $F_\lambda(z)$ <u>be in</u> $\bar{G}^*(\alpha, \beta, \lambda)$ <u>and let</u> r_0, r_1 and r_2 <u>be the same as in Theorem</u> 5.5.1. <u>Then</u> (i) F_λ <u>is convex in</u> $|z| < r_1$ <u>for</u> $0 < r \leq r_0$ <u>and</u> (ii) F_λ <u>is convex in</u> $|z| < r_2$ <u>for</u> $r_0 \leq r < 1$. <u>The results are sharp</u>.

<u>Proof</u> : Since $F_\lambda \in \bar{G}^*(\alpha, \beta, \lambda)$, (5.1.6) gives

$$z F_\lambda'(z) = (1-\lambda)(zf'(z)) + \lambda z(zf'(z))' \tag{5.5.10}$$

where $f \in C(\alpha, \beta)$. Thus by Remark 2.1.2 and Theorem 5.5.1, it follows that zF_λ' is starlike in $|z| < r_1$ for $0 < r \leq r_0$ and is starlike in $|z| < r_2$ for $r_0 \leq r < 1$ where r_0, r_1 and r_2 are same as in Theorem 5.5.1.

Again, applying Remark 2.1.2, the theorem follows.

The functions given by

$$(5.5.11) \qquad 1 + z\frac{f''(z)}{f'(z)} = \frac{1-(2\alpha\beta-1)z}{1-(2\beta-1)\,z}$$

and

$$(5.5.12) \qquad 1 + z\frac{f''(z)}{f'(z)} = \frac{1-2\alpha\beta bz+(2\alpha\beta-1)z^2}{1-2\beta bz+(2\beta-1)z^2}$$

where b is determined by the relation (5.5.8), show that the results contained in the theorem are sharp.

Remark 5.5.2 : (a) A result similar to Nikolaeva and Repnine [56] can be derived from Theorem 5.5.2, by putting $\beta = 1$.

(b) Putting $\lambda = 1/2$ and $(\alpha,\beta) = (0,1)$ in Theorem 5.5.2, we get the result obtained by Livingston [39].

5.6 The radii of starlikeness for functions in $H_k^*(\alpha,\delta,\lambda)$ and $\bar{H}_k^*(\gamma,\delta,\lambda)$.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in Δ and $g \in S$. MacGregor [45] determined the radius of univalence of the class H of functions $f(z)$ satisfying $|f(z)/g(z) - 1| < 1$ for $z \in \Delta$. He also obtained the radius of univalence and starlikeness for the subclasses of H when $g \in S^*$ or C. Ratti [70], then obtained the radius of univalence and starlikeness of $f(z)$ when g is starlike of order $\alpha (0 \le \alpha < 1)$ in Δ. Later, Padmanabhan [59,60] considered the classes of functions $f(z)$ satisfying

$$(5.6.1) \qquad \left|\frac{f(z)}{g(z)} - 1\right| < \delta, \quad 0 < \delta \le 1$$

where $g \in S^*$ or S^*_α respectively and determined the radius of univalence and starlikeness for functions in these classes. Recently, Shah [77] considered the class $H^*_k(\alpha,1,\lambda)$ of analytic functions $f(z)$, given by (5.1.1), and satisfying

$$(5.6.2) \qquad \left|\frac{f(z)}{\lambda f(z) + (1-\lambda)g(z)} - 1\right| < 1$$

for $z \in \Delta$, $0 \le \lambda < 1$ and $g \in S^*_{\alpha,k}$. He determined the disc in which f is univalent and starlike. However, the results obtained by Shah [77] are not sharp.

In this section, we first determine the radii of starlikeness for the class $H^*(\gamma,\delta)$ of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in Δ and satisfying (5.6.1) for some $g \in S^*(\gamma)$. We then generalize the results of Shah [77] by determining radii of starlikeness for functions in $H^*_k(\alpha,\delta,\lambda)$. We also obtain the radii of starlikeness for functions in $\bar{H}^*_k(\gamma,\delta,\lambda)$. Thus we have the following results :

Theorem 5.6.1 : Let $f(z)$ be in $H^*(\gamma,\delta)$ and let $\delta_0(\gamma)=1/4((1-2\gamma)+\sqrt{(4\gamma+5)})$. Further, let $U_1 = \{(\gamma,\delta) : 0 < \gamma \le 1,\ 0 < \delta \le 1\}$, $I_1=\{(\gamma,\delta) : 0 < \gamma \le 1,\ \delta_0(\gamma) \le \delta \le 1\}$, $I_2 = G-I_1$ where $G = \{(\gamma,\delta) : 0 < \gamma \le 1,\ 1/2 \le \delta \le 1\}$ and $I_3 = U_1-(I_1 \cup I_2)$. Then

(i) f is starlike in $|z| < r_1$ for $(\gamma,\delta) \in I_1$ and

(ii) f is starlike in $|z| < r_2$ for $(\gamma,\delta) \in I_2 \cup I_3$ where

$$r_1 = (\gamma + 2\delta)^{-1}$$

<u>and</u> r_2 <u>is the smallest positive root of the equation</u>

$$(\delta-\gamma^2+\delta\gamma^2)r^4+2\gamma\delta\, r^3 + (1-\delta+\gamma^2)r^2-(1-\delta) = 0.$$

<u>The estimate</u> (i) <u>is sharp</u>.

<u>Proof</u> : Since $f \in H^*(\gamma,\delta)$, we have

$$(5.6.3) \qquad \left|\frac{f(z)}{g(z)} - 1\right| < \delta,\ 0 < \delta \le 1$$

for some $g \in S^*(\gamma)$ and $0 \le \gamma < 1$, $z \in \Delta$. Thus, by Schwarz's Lemma, we have

$$(5.6.4) \qquad f(z) = (1-\delta\omega(z))g(z)$$

where $\omega \in B$. Differentiating (5.6.4) logarithmically, we have

$$(5.6.5) \qquad z\frac{f'(z)}{f(z)} = z\frac{g'(z)}{g(z)} - \delta\left\{\frac{z\omega'(z)}{1-\delta\omega(z)}\right\}.$$

Applying (3.2.2) with $k = 1$, $s = 0$, $t = -\delta$ and using (2.3.5) with $\alpha = (1-\gamma)/(1+\gamma)$ and β by $(1+\gamma)/2$ in (5.6.5), we obtain

$$(5.6.6) \qquad \mathrm{Re}\left\{z\frac{f'(z)}{f(z)}\right\} \ge \frac{2}{1+\gamma r} + \frac{1}{\delta}\left[\mathrm{Re}\,\frac{(-\delta)}{p(z)} - \frac{\delta^2 r^2-|1-p(z)|^2}{(1-r^2)|p(z)|}\right].$$

An application of Lemma 3.2.3 with $k = 1$, $q = s = 0$ and $t = -\delta$ to (5.6.6) gives

$$(5.6.7) \quad \mathrm{Re}\left\{z\frac{f'(z)}{f(z)}\right\} \ge \begin{cases} \dfrac{2}{\delta(1-r^2)(1+\gamma r)}\left[(1+\gamma r)\{\sqrt{(1-\delta)(1+\delta r^2)}-1\}+\delta(1-r^2)\right] & \text{if } R_0 \ge R_1, \\[2ex] \dfrac{1-(\gamma+2\delta)r}{(1+\gamma r)(1-\delta r)} & \text{if } R_0 \le R_1 \end{cases}$$

where $R_o^2 = (1-\delta)(1+\delta r^2)$ and $R_1 = (1-\delta r)$.

Now proceeding on the similar lines as in Theorem 3.3.1, the required results follow easily.

Figure 5 shows the transition curve C_o and the regions I_1, I_2 and I_3.

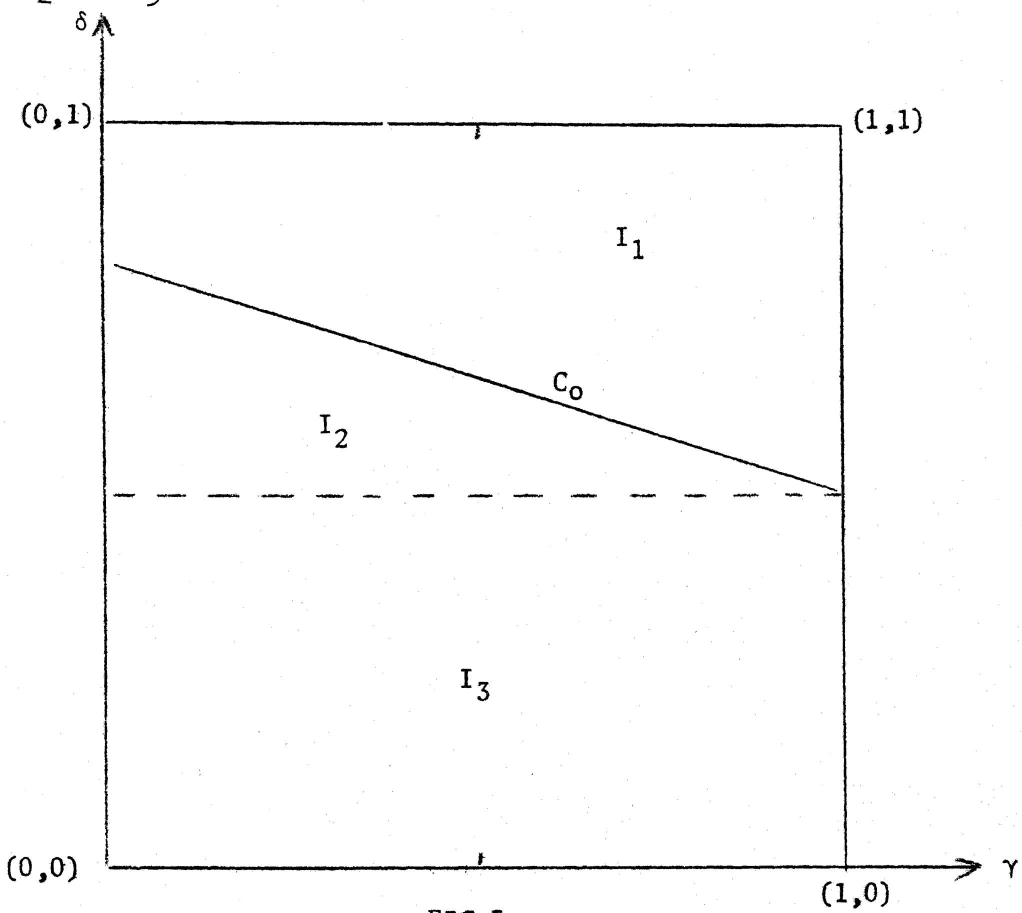

FIG-5

The estimate (i) is sharp for the functions given by

$$f(z) = (1-\delta z)\, g(z)\ ;\ g(z) = \frac{z}{(1-\gamma z)^2}\ .$$

Remark 5.6.1 : Putting $(\gamma,\delta) = (1,1)$, we obtain the corresponding result obtained by MacGregor [45] .

Theorem 5.6.2 : Let $f(z)$ be in $H_k^*(\alpha,\delta,\lambda)$ for $0 \leq \lambda \leq 1/(1+\delta)$, $0 < \delta \leq 1$, let r_o be the smallest positive root, which exists in $(0,1)$ of the equation

$$(5.6.8)\quad \lambda k\delta^2 r^{2k+2} - 2\lambda\delta^2 r^{2k+1} - \lambda k\delta^2 r^{2k} + 2\delta(2\lambda-1)r^{k+1} + k(1-\lambda)r^2 + 2(1-\lambda)r - k(1-\lambda) = 0;$$

then f is starlike for $|z| < r_1$, where r_1 is the smallest positive root of the equation

$$(1-2\alpha)\lambda\delta^2 r^{3k} + \delta((1-k)-2\alpha(1-2\lambda)-\lambda(2+\delta))r^{2k} + (-1-\delta(1+k)+\lambda(1+2\delta)+2\alpha(1-\lambda))r^k + (1-\lambda) = 0$$

if $0 < r \leq r_o$ and f is starlike for $|z| < r_2$, where r_2 is the smallest positive root of the equation

$$[4(1-\lambda)+8\alpha(1-\lambda)r^k + 4((1-\lambda)(2\alpha-1)+\lambda\delta^2(4\lambda k-2k-1))r^{2k} - 4\lambda\delta^2(\alpha+2k-4\lambda k)r^{3k} - 4\lambda\delta^2(2\alpha-1+2k-4\lambda k)r^{4k}](1-r^2) - [\delta((k+1)^2-4\lambda k)r^{k-1} + 2\delta(k^2-1+2\alpha(2+k)-4\alpha\lambda k)r^{2k-1} + \delta((2\alpha-1)^2+k^2+2k(2\alpha-1)(1-2\lambda))r^{3k-1}](1+r^4) + 2(2\delta k(1-2\lambda) + 8\lambda\delta(1-\lambda) - \delta(1-k^2))r^{k+1} + 2\delta(k^2-2-2k+(2\alpha-1)^2+8\lambda(1-\lambda)+4\lambda k(1-2\alpha) + 4\alpha k)\, r^{3k+1} = 0$$

if $r_o \leq r < 1$. The estimate in Ist part of the theorem is sharp for the functions given by

$$(5.6.9)\qquad f(z) = (1-\delta z^k)\, g(z)\ ;\ g(z) = \frac{z}{(1-z^k)^{2(1-\alpha)/k}}.$$

Proof : Since $f \in H_k^*(\alpha,\delta,\lambda)$, we have

$$(5.6.10) \qquad \left| \frac{f(z)}{\lambda f(z) + (1-\lambda)g(z)} - 1 \right| < \delta$$

for some δ, $0 < \delta \leq 1$, $g \in S^*_{\alpha,k}$ and $0 \leq \lambda < 1$. Thus, by Schwarz's Lemma

$$(5.6.11) \qquad f(z) = \frac{1-\delta z^{k-1}\ \omega(z)}{1 + \tau z^{k-1}\ \omega(z)}\, g(z)$$

where $\tau = \lambda\delta/(1-\lambda)$ and $\omega \in \mathcal{B}$. Differentiating (5.6.11) logarithmically, we have

$$(5.6.12) \qquad z\,\frac{f'(z)}{f(z)} = z\,\frac{g'(z)}{g(z)} - \frac{\delta}{1-\lambda}\left\{\frac{z^k\ \omega'(z)+(k-1)z^{k-1}\omega(z)}{(1-\delta z^{k-1}\ \omega(z))(1+\tau z^{k-1}\ \omega(z))}\right\}.$$

Applying (3.2.2) with $s = \tau$ and $t = -\delta$ and using (2.3.5) with $\beta = 1$ in (5.6.12), we get

$$(5.6.13) \qquad \operatorname{Re}\left\{z\,\frac{f'(z)}{f(z)}\right\} \geq (1-2\lambda)k + \frac{1+(2\alpha-1)r^k}{1+r^k} + \frac{1-\lambda}{\delta}\Big[\operatorname{Re}\left\{k\tau\, p(z) + \frac{(-\delta)k}{p(z)}\right\} - \frac{r^{2k}|\tau\, p(z)+\delta|^2-|1-p(z)|^2}{r^{k-1}\ (1-r^2)|p(z)|}\Big]$$

where $p(z) = (1-\delta z^{k-1}\ \omega(z))/(1+\tau z^{k-1}\ \omega(z))$.

An application of Lemma 3.2.3 with $q = \tau k$, $s = \tau$ and $t = -\delta$ to (5.6.13) gives

$$(5.6.14)\quad \mathrm{Re}\left\{z\frac{f'(z)}{f(z)}\right\} > \begin{cases} \dfrac{1}{\delta r^{k-1}(1-r^2)(1+r^k)}\,[2(1+r^k)\sqrt{\bar{N}} - 2(1-\lambda-\lambda\delta^2 r^{2k})(1+r^k) + \\ \quad +(2\alpha-1+k(1-2\lambda))r^k + \delta r^{k-1}(1+(1-2\lambda)k)(1-r^2)\,] \\ \hfill \text{if } R_o \geq R_1, \\ \dfrac{(1-2\alpha)\lambda\delta^2 r^{3k} + \delta((1-k)-2(1-2\lambda)\alpha-\lambda(2+\delta))r^{2k} + (-1-\delta(1+k)+\lambda(1+2\delta)+2\alpha(1-\lambda))r^k + (1-\lambda)}{(1+r^k)(1-\delta r^k)(1-\lambda+\lambda\delta r^k)} \\ \hfill \text{if } R_o \leq R_1 \end{cases}$$

where

$$\bar{N} = (1-\delta k r^{k-1}(1-r^2)-\delta^2 r^{2k})((1-\lambda)^2+\delta\lambda k(1-\lambda)r^{k-1}(1-r^2)-\lambda^2\delta^2 r^{2k}),$$

$$R_o^2 = \frac{1-\delta k\ r^{k-1}(1-r^2)-\delta^2 r^{2k}}{1+\tau k\ r^{k-1}(1-r^2)-\tau^2 r^{2k}} \quad \text{and } R_1 = \frac{1-\delta r^k}{(1+\tau r^k)}\,.$$

The result now follows easily from (5.6.14) by proceeding on the similar lines as in Theorem 5.2.1.

Remark 5.6.1 : (a) A result similar to Shah [77] for the range $0 \leq \lambda \leq 1/2$ can be deduced from Theorem 5.6.2, by putting $\delta = 1$. It can be noted that our estimates are sharper than those obtained by Shah [77].

(b) For $k = 1$ and $\lambda = 0$ in Theorem 5.6.2, we get the corresponding result obtained by Padmanabhan [60].

Putting $\lambda = 0$, $k = 1$ and $\delta = 1$ in Theorem 5.6.2, we obtain the following result due to Ratti [70] which also generalizes the corresponding result due to MacGregor [45].

Corollary 5.6.2 : Let $f(z) = z + a_2 z^2 + \ldots$ and $g(z) = z + b_2 z^2 + \ldots$ be analytic in Δ and $g \in S_\alpha^*$ in Δ. If $|f(z)/g(z)-1| < 1$ for $z \in \Delta$, then f is univalent and starlike for $|z| < r'$, where $r'=((2\alpha-3)+\sqrt{(9-4\alpha+4\alpha^2)})/4\alpha$ if $\alpha \neq 0$ and $r' = 1/3$ if $\alpha = 0$. The result is sharp for the functions given by (5.6.9) with $k = 1$, $\delta = 1$.

The proof of the following theorem is similar to the above theorem hence omitted.

Theorem 5.6.3 : Let f(z) be in $\bar{H}_k^*(\gamma,\delta,\lambda)$ for $0 \leq \lambda \leq 1/(1+\delta)$, $0 < \delta \leq 1$, let r_o be the smallest positive root, which exists in (0,1) of the equation (5.6.8); then f is starlike for $|z| < r_1$, where r_1 is the smallest positive root of the equation

$$\lambda\gamma\delta^2 r^{3k}+\delta(\gamma(1-k)-\lambda(\delta+2\gamma))r^{2k}-(\gamma(1-\lambda)+\delta(1+k)-2\lambda\delta)r^k + (1-\lambda) = 0$$

if $0 < r \leq r_o$ and f is starlike in $|z| < r_2$, where r_2 is the smallest positive root of the equation

$$\begin{aligned}&4[(1-\lambda)+(1-\lambda)(1-\gamma)(k-1-2\lambda k)r^k+(\lambda(4\lambda k-2k-1)\delta^2-\gamma(1-\lambda)+\gamma k(1-\gamma)(1-3\lambda+2\lambda^2))r^{2k}\\&-\lambda\delta^2((\gamma-1)+k(1-2\lambda)(1+\gamma))r^{3k}+\lambda\gamma\delta^2(1-k(1+\gamma)(1-2\lambda))r^{4k}](1-r^2)+[\delta(4\lambda k -\\&-(k+1)^2)r^{k-1} + 2\delta((1-k^2)+4\lambda(1-\lambda)(1-\gamma)k^2)r^{2k-1}-\delta((1-2k+k^2)+4\lambda k(1-k+\lambda k)+\\&+4\lambda k^2\gamma^2(1-\lambda))r^{3k-1}](1+r^4)-2\delta((1-2k-k^2)-8\lambda(1-\lambda)+4\lambda k)r^{k+1}+2\delta((1-k)^2+\\&+4\lambda k-2\gamma^2(1-2\lambda)^2+4\lambda^2k^2(1-\gamma^2))r^{3k+1} = 0\end{aligned}$$

if $r_o \leq r < 1$. The estimate in Ist part of the theorem in sharp for the functions given by

$$f(z) = (1-\delta z^k)\ g(z)\ ;\ g(z) = \frac{z}{(1-\gamma z^k)^{2/k}} .$$

Remark 5.6.3 : $\lambda = 0$, $k = 1$ and $(\gamma,\delta) = (1,1)$ in Theorem 5.6.3 leads to the corresponding result obtained by MacGregor [45].

The following results can easily be deduced from the above two theorems by making use of the fact that $f \in C_k(\alpha,\beta)$ if, and only if, $zf' \in S^*_k(\alpha,\beta)$.

Theorem 5.6.4 : Let $f(z)$ be in $Q^*_k(\alpha,\delta,\lambda)$ for $0 \leq \lambda \leq 1/(1+\delta), 0 < \delta \leq 1$ and let r_o, r_1 and r_2 be the same as in Theorem 5.6.2. Then (i) f is convex in $|z| < r_1$ for $0 < r \leq r_o$ and (ii) f is convex in $|z| < r_2$ for $r_o \leq r < 1$. The estimate (i) is sharp for the functions given by

$$f'(z) = (1-\delta z^k)\ g'(z);\ g'(z) = \frac{1}{(1-z^k)^{2(1-\alpha)/k}} .$$

Theorem 5.6.5 : Let $f(z)$ be in $\bar{Q}^*_k(\gamma,\delta,\lambda)$ for $0 \leq \lambda \leq 1/(1+\delta)$, $0 < \delta \leq 1$ and let r_o, r_1 and r_2 be the same as in Theorem 5.6.3. Then (i) f is convex in $|z| < r_1$ for $0 < r \leq r_o$ and (ii) f is convex in $|z| < r_2$ for $r_o \leq r < 1$. The estimate (i) is sharp for the functions given by

$$f'(z) = (1-\delta z^k)\ g'(z)\ ;\ g'(z) = \frac{1}{(1-\gamma z^k)^{2/k}} .$$

CHAPTER VI

EFFECT OF SECOND COEFFICIENT ON CERTAIN PROPERTIES OF SUBCLASSES OF UNIVALENT FUNCTIONS

6.1 Let

$$f(z) = z + a_2 z^2 + \ldots..$$

be analytic and univalent in Δ. The effect of second coefficient on various properties of $f \in S$ has been studied by several workers. Thus, in 1920, Gronwall [22] obtained growth theorems, involving the second coefficient, for the class of convex functions. Finkelstein [15] and Tepper [81] studied the effect of second coefficient on certain properties, e.g., distortion theorems, radii of convexity etc. of starlike and convex functions. Later, Padmanabhan [62], McCarty [51 ,52] and others extended this type of study by considering starlike and convex functions of order $\alpha(0 \leq \alpha < 1)$. McCarty [51 ,52] also obtained distortion theorems, radii of convexity etc., for the class R_α of functions with fixed second coefficient, whose derivatives have a real part greater than $\alpha(0 \leq \alpha < 1)$ in Δ. Recently, the radius of starlikeness and convexity, involving second coefficient, for the class of functions of the form $1/2(zf(z))'$ where $f \in S^*$ has been obtained by Al-Amiri [1].

For a unified study of the various classes of functions with fixed second coefficient, we first introduce the class $P(\alpha,\beta)$ as follows.

Definition 6.1.1: Let $p(z) = 1 + b_1 z + \ldots$ be analytic in Δ. Then $p \in P(\alpha,\beta)$ if

$$(6.1.1) \qquad |(p(z)-1)/\{2\beta(p(z)-\alpha)-(p(z)-1)\}| < 1$$

holds for some α, β $(0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ and all $z \in \Delta$.

It can be easily seen that $|b_1| \leq 2(1-\alpha)\beta$. Further, we note that if $\varepsilon = \exp\{-i \arg b_1\}$, then $p(\varepsilon z) = 1 + |b_1|z + \ldots$ and so, to limit our study of $P(\alpha,\beta)$ to functions with a non-negative first real coefficient is actually no restriction. Hence, throughout this chapter, we shall take the fixed coefficient to be real and positive.

Now we define the following classes:

$$P(b;\alpha,\beta) = \{p \in P(\alpha,\beta):\ p'(0) = 2b(1-\alpha)\beta, 0 \leq b \leq 1\}$$

$$R(a;\alpha,\beta) = \{f(z) = z + a\beta(1-\alpha)z^2 + \ldots:\ f' \in P(a;\alpha,\beta), 0 \leq a \leq 1\}$$

$$S^*(a;\alpha,\beta) = \{g(z) = z + 2a\beta(1-\alpha)z^2 + \ldots : zg'/g \in P(a;\alpha,\beta), 0 \leq a \leq 1\}$$

$$V^*(a;\alpha,\beta) = \{F(z) = 1/2\ (zf(z))':\ f \in S^*(a;\alpha,\beta)\}.$$

In the present chapter, we first determine growth theorems for functions in the classes $P(b;\alpha,\beta)$ and $R(a;\alpha,\beta)$. We then obtain sharp estimates for the radii of convexity for functions in $R(a;\alpha,\beta)$ and $S^*(a;\alpha,\beta)$. Finally, we determine the exact radii of starlikeness for functions in the class $V^*(a;\alpha,\beta)$. It will be seen that, for different values of the parameters α and β, our results not only yield the results obtained earlier by Tepper [81], Goel [20], McCarty [51, 52], Al-Amiri [1] etc. but also give rise to analogous

results for a number of other subclasses of univalent functions.

6.2 Growth theorems for functions in $P(b;\alpha,\beta)$ and $R(a;\alpha,\beta)$.

We require the following lemmas.

Lemma 6.2.1: *Let* $\omega(z) = a_1 z + \dots$ *be an analytic function mapping the unit disc onto* into *itself. Then* $|a_1| \leq 1$ *and*

$$|\omega(z)| \leq \frac{r(r+|a_1|)}{(1+|a_1|r)}, \quad |z| = r.$$

The above Lemma is the iterated form of Schwarz's Lemma and is due to Löwner [40] .

Lemma 6.2.2: *Let* $\omega(z) = a_1 z + \dots$ *be analytic for* $|z| < 1$ *and map the unit disc onto itself. Then the values assumed by the function* $H(z) = (1+(2\alpha\beta-1)\,\omega(z))/(1+(2\beta-1)\omega(z))$ *for* $|z| < r < 1$ *lie within the circle whose diameter is the straight line segment joining the points*

$$\frac{1+2\alpha\beta \overset{|a_1|}{b} r + (2\alpha\beta-1)r^2}{1+2\beta \underset{|a_1|}{b} r + (2\beta-1)\, r^2} \quad \text{and} \quad \frac{1+2(1-\alpha\beta)br+(1-2\alpha\beta)r^2}{1+2(1-\beta)br + (1-2\beta)r^2}.$$

Proof: It is easily seen that the function $(1+(2\alpha\beta-1)\zeta)/(1+(2\beta-1)\zeta)$ with $0 \leq \alpha < 1$, $0 < \beta \leq 1$, maps $|\zeta| < r' < 1$ onto a disc whose diameter is a straight line segment joining the points $(1+(2\alpha\beta-1)r')/(1+(2\beta-1)r')$ and $(1-(2\alpha\beta-1)r')/(1-(2\beta-1)r')$ of the real axis. The function $\omega(z)$ satisfies the conditions of Lemma 6.2.1 and so

$$|\omega(z)| \leq r(r+|a_1|)/(1+|a_1|r), \quad |z| = r.$$

Setting $\zeta = \omega(z)$, we note that for $|z| < r$, we have $|\zeta| < r'$ where $r' = r(r+|a_1|)/(1+|a_1|r)$. Hence, the values assumed by $H(z) = (1+(2\alpha\beta-1)\omega(z))/(1+(2\beta-1)\omega(z))$ in $|z| < r$ are among the values assumed by $(1+(2\alpha\beta-1)\zeta)/(1+(2\beta-1)\zeta)$ in $|\zeta| < r'$. Thus the conclusion of the lemma follows.

<u>Theorem 6.2.1</u>: <u>If</u> $p \in P(b;\alpha,\beta)$, <u>then for</u> $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $z \in \Delta$

$$(6.2.1) \qquad |p(z)| \leq \frac{1+2(1-\alpha\beta)b\,|z| + (1-2\alpha\beta)|z|^2}{1+2(1-\beta)b\,|z| - (2\beta-1)|z|^2}$$

$$(6.2.2) \qquad \mathrm{Re}\ \{p(z)\} \geq \frac{1+2\alpha\beta b|z| + (2\alpha\beta-1)|z|^2}{1+2\beta b|z| + (2\beta-1)|z|^2}.$$

<u>The estimates are sharp</u>.

<u>Proof</u>: Since $p \in P(b;\alpha,\beta)$, by Schwarz's Lemma, we have

$$(6.2.3) \qquad p(z) = \frac{1+(2\alpha\beta-1)\omega(z)}{1+(2\beta-1)\,\omega(z)}$$

where $\omega \in B$. (6.2.3) gives

$$\omega(z) = \frac{1-p(z)}{2\beta(p(z)-\alpha)-(p(z)-1)} = -[bz + \ldots.] .$$

Therefore by Lemma 6.2.2, $p(z)$ assumes values lying in the disk K, On the line segment joining the points $(1+2\alpha\beta br+(2\alpha\beta-1)r^2)/(1+2\beta br+(2\beta-1)r^2)$ and $(1+2(1-\alpha\beta)br + (1-2\alpha\beta)r^2)/(1+2(1-\beta)br + (1-2\beta)r^2)$ as diameter. Thus, If $h(z) = (1+2(1-\alpha\beta)bz + (1-2\alpha\beta)z^2)/(1+2(1-\beta)bz + (1-2\beta)z^2)$ then, since

$h(0) = p(0) = 1$ and h is univalent in Δ, it follows that p is subordinate to h. Hence we have

$$(6.2.4) \qquad |p(z)| \leq \frac{1+2(1-\alpha\beta)b\,|z|+(1-2\alpha\beta)\,|z|^2}{1+2(1-\beta)b\,|z|-(2\beta-1)\,|z|^2}$$

and

$$(6.2.5) \qquad \frac{1+2\,\alpha\beta b\,|z|+(2\alpha\beta-1)\,|z|^2}{1+2\beta b\,|z|+(2\beta-1)\,|z|^2} \leq \mathrm{Re}(p(z)) \leq \frac{1+2(1-\alpha\beta)b\,|z|+(1-2\alpha\beta)\,|z|^2}{1+2(1-\beta)b\,|z|-(2\beta-1)\,|z|^2}.$$

Thus the theorem follows from (6.2.4) and (6.2.5).

Equality in (6.2.1) and (6.2.2) holds for the functions

$$p_1(z) = \frac{1+2(1-\alpha\beta)bz + (1-2\,\alpha\beta)z^2}{1+2(1-\beta)bz-(2\beta-1)\,z^2}$$

and

$$p_2(z) = \frac{1-2\,\alpha\beta bz + (2\alpha\beta-1)z^2}{1-2\beta\, bz + (2\beta-1)\,z^2}\,.$$

Putting $\beta = 1$ in Theorem 6.2.1, we get the following growth theorems for functions in $P(b;\alpha,1)$ due to McCarty [51].

<u>Corollary 6.2.1a</u> : <u>If</u> $p \in P(b;\alpha,1)$, <u>then for</u> $0 \leq \alpha < 1$, $z \in \Delta$

$$|p(z)| \leq \frac{1+2(1-\alpha)b\,|z|+(1-2\alpha)\,|z|^2}{1-|z|^2}$$

$$\mathrm{Re}\,(p(z)) \geq \frac{1+2\alpha b\,|z|+(2\alpha-1)\,|z|^2}{1+2b\,|z|+|z|^2}\,.$$

<u>The estimates are sharp</u>.

<u>Corollary 6.2.1b</u> [81] : <u>If</u> $p \in P(b;0,1)$, <u>then for</u> $z \in \Delta$

$$|p(z)| \leq \frac{1 + 2b|z| + |z|^2}{1 - |z|^2}$$

$$\mathrm{Re}(p(z)) \geq \frac{1 - |z|^2}{1 + 2b|z|+|z|^2} .$$

<u>The inequalities are sharp</u>.

This result is obtained by taking $(\alpha, \beta) = (0,1)$ in Theorem 6.2.1 and is due to Tepper [81].

<u>Theorem 6.2.2</u> : <u>If</u> $f \in R(a;\alpha,\beta)$, <u>then for</u> $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $z \in \Delta$

$$(6.2.5) \qquad |f'(z)| \leq \frac{1 + 2(1-\alpha\beta)a|z| + (1-2\alpha\beta)|z|^2}{1 + 2(1-\beta)a|z|-(2\beta-1)|z|^2}$$

$$(6.2.6) \qquad \mathrm{Re}\{f'(z)\} \geq \frac{1 + 2\alpha\beta a|z| + (2\alpha\beta-1)|z|^2}{1 + 2a\beta|z| + (2\beta-1)|z|^2} .$$

<u>The estimates obtained in the theorem are sharp</u>.

<u>Proof</u> : Theorem 6.2.2 immediately follows from Theorem 6.2.1 by putting $p(z) = f'(z)$.

The results in (6.2.5) and (6.2.6) are sharp for the functions given by

$$(6.2.7) \qquad f_1(z) = \int_0^z \frac{1 + 2(1-\alpha\beta)at + (1-2\alpha\beta)t^2}{1 + 2(1-\beta)at - (2\beta-1)t^2}\, dt$$

and

$$(6.2.8) \qquad f_2(z) = \int_0^z \frac{1 - 2\alpha\beta at + (2\alpha\beta-1)t^2}{1-2a\beta t + (2\beta-1)t^2}\, dt$$

for $z = |z|$ and $z = -|z|$ respectively.

Putting $\beta = 1$ in Theorem 6.2.1, we get the following growth theorems due to McCarty [51] .

Corollary 6.2.2a : If $f \in R(a;\alpha,1)$, then for $0 \leq \alpha < 1$, $z \in \Delta$

$$|f'(z)| \leq \frac{1 + 2a(1-\alpha)|z| + (1-2\alpha)\ |z|^2}{1-|z|^2}$$

$$\text{Re}\ (f'(z)) \geq \frac{1 + 2a\alpha|z| + (2\alpha-1)|z|^2}{1 + 2a|z| + |z|^2}.$$

The estimates are sharp for the functions given by (6.2.7) and (6.2.8) with $\beta = 1$.

Corollary 6.2.2b [20] : If $f \in R\ (a;(1-\alpha),1/2)$, then for $z \in \Delta$

$$|f'(z)| \leq \frac{1 + (1+\alpha)a|z| + \alpha|z|^2}{1 + a|z|}$$

$$\text{Re}(f'(z)) \geq \frac{1 + (1-\alpha)a|z| - \alpha|z|^2}{1 + a|z|}.$$

The estimates are sharp.

This result is obtained by replacing α by $1-\alpha$ and β by 1/2 in Theorem 6.2.2 and was obtained by Goel [20].

Corollary 6.2.2c : If $f(z) = z + a\gamma z^2 + \ldots$ such that $f \in R\ (\gamma)$, then for $0 < \gamma \leq 1$, $z \in \Delta$

$$|f'(z)| \leq \frac{1 + (1+\gamma)a|z| + \gamma|z|^2}{1 + (1-\gamma)a|z| - \gamma|z|^2}$$

$$\text{Re}\ (f'(z)) \geq \frac{1 + (1-\gamma)a\ |z| - \gamma|z|^2}{1 + (1+\gamma)a\ |z| + \gamma|z|^2}.$$

The estimates are sharp.

This result is obtained by replacing α by $(1-\gamma)/(1+\gamma)$ and β by $(1+\gamma)/2$ in Theorem 6.2.2.

6.3 The radii of convexity for functions in $R(a;\alpha,\beta)$ and $S^*(a;\alpha,\beta)$.

To prove the main results, we require the following lemmas.

Lemma 6.3.1 : If $p(z) = (1+t\omega(z))/(1+s\omega(z))$, $\omega \in B$, then $|p(z)-A_b| \leq D_b$ for $0 \leq b \leq 1$, where

$$A_b = \frac{(1+br)^2-str^2(b+r)^2}{(1+br)^2-s^2r^2(b+r)^2} \; ; \quad D_b = \frac{(s-t)r(b+r)(1+br)}{(1+br)^2-s^2r^2(b+r)^2}$$

and $z \in \Delta$, $-1 \leq t < s \leq 1$.

Proof : Since $p(z) = (1+t\omega(z))/(1+s\omega(z))$, we have

$$\omega(z) = \frac{1-p(z)}{sp(z)-t} = -[bz + \ldots] = -z\phi(z) \tag{6.3.1}$$

where ϕ is analytic and $|\phi(z)| \leq 1$ for $z \in \Delta$ with $\phi'(0) = b$. Now, since $(\phi(z)-b)/(1-b\phi(z))$ is subordinate to z, it follows that $\phi(z)$ is subordinate to $(z+b)/(1+bz)$ and

$$|\omega(z)| = \left|\frac{1-p(z)}{sp(z)-t}\right| = |z\phi(z)| \leq |z| \frac{(|z|+b)}{(1+b|z|)} .$$

Thus, we have

$$\left|\frac{1-p(z)}{sp(z)-t}\right| \leq |z| \frac{(|z|+b)}{(1+b|z|)} . \tag{6.3.2}$$

Let $p(z) = \xi + i\eta$. On simplification, (6.3.2) gives

$$\left(\xi - \frac{(1+br)^2 - str^2(b+r)^2}{(1+br)^2 - s^2r^2(b+r)^2}\right)^2 + \eta^2 \leq \frac{(s-t)^2 r^2 (b+r)^2 (1+br)^2}{((1+br)^2 - s^2r^2(b+r)^2)^2}$$

i.e.,

$$\left|\xi + i\eta - \frac{(1+br)^2 - str^2(b+r)^2}{(1+br)^2 - s^2r^2(b+r)^2}\right| \leq \frac{(s-t)r(b+r)(1+br)}{(1+br)^2 - s^2r^2(b+r)^2}.$$

Hence $|p(z) - A_b| \leq D_b$, where A_b and D_b are same as defined in the lemma.

<u>Lemma 6.3.2</u> : <u>If</u> $p(z) = (1+t\omega(z))/(1+s\omega(z))$, $\omega \in \mathcal{B}$, <u>then for</u> $|z| = r$, $0 \leq r < 1$, <u>we have</u>

$$\mathrm{Re}\{kp(z) + \frac{t}{p(z)}\} - \frac{r^2|sp(z) - t|^2 - |1 - p(z)|^2}{(1-r^2)|p(z)|}$$

$$(6.3.3) \qquad \geq \begin{cases} \dfrac{2}{1-r^2}\left[\sqrt{(1+t)(1-tr^2)(1+k(1-r^2)-s^2r^2)} - (1-str^2)\right] & \text{if } R_b \leq R', \\ W/W^* & \text{if } R_b \geq R' \end{cases}$$

<u>where</u>

$$(6.3.3;a) \quad W = (1+k-kr^2-s^2r^2)((1+br)^2 - str^2(b+r)^2 - (s-t)r(b+r)(1+br))^2 + (1+t-tr^2-t^2r^2) \times ((1+br)^2 - s^2r^2(b+r)^2) - 2(1-str^2)((1+br)^2 - s^2r^2(b+r)^2)((1+br)^2 - str^2(b+r)^2 - (s-t)r(b+r)(1+br)),$$

$$(6.3.3;b) \quad W^* = (1-r^2)((1+br)^2 - s^2r^2(b+r)^2)((1+br)^2 - str^2(b+r)^2 - (s-t)r(b+r)(1+br)),$$

<u>and</u> $R'^2 = (1+t)(1-tr^2)/(k(1-r^2)+1-s^2r^2)$, $R_b = A_b - D_b$ <u>where</u> A_b, D_b <u>are defined as in</u> Lemma 6.3.1 <u>and</u> $k \geq s$, $-1 \leq t < s \leq 1$.

__Proof :__ Let $p(z) = A_b + \xi + i\eta$ and $R^2 = (A_b+\xi)^2+\eta^2$ with $|z| = r$. Denoting the left hand side of (6.3.3) by $U_b(\xi,\eta)$, then

$$(6.3.4)\quad U_b(\xi,\eta) = k(A_b+\xi)+t(A_b+\xi)R^{-2} + \frac{1-s^2r^2}{1-r^2}\,[\{(A_b+\xi)-A_1\}^2 + \eta^2-D_1^2]^{-1}$$

and

$$(6.3.5)\qquad \frac{\partial U_b}{\partial \eta} = \eta R^{-4}\, V_b(\xi,\eta)$$

where

$$(6.3.6)\quad V_b(\xi,\eta) = -2t(A_b+\xi)+(D_1^2+2A_1(A_b+\xi)-A_1^2)(\frac{1-s^2r^2}{1-r^2})R + \frac{1-s^2r^2}{1-r^2}\,R^3.$$

Denote by $F_b(R)$ the right hand side of (6.3.6) with $R\cos\psi = A_b+\xi$, then

$$F_b(R)=-2tR\cos\psi + (D_1^2-A_1^2+2A_1\cos\psi R)\,(\frac{1-s^2r^2}{1-r^2})\,R + (\frac{1-s^2r^2}{1-r^2})\,R^3$$

$$= 2\,[A_1R(\frac{1-s^2r^2}{1-r^2})-t]\,R\cos\psi + (D_1^2-A_1^2+R^2)\,(\frac{1-s^2r^2}{1-r^2})R.$$

Geometrical considerations show that for $R \in [A_b - D_b,\ A_b + D_b]$ the function $F_b(R)$ increases with increasing R. Since $R\cos\psi$ is the projection onto the real axis, it must lie on the diameter of the circle of Lemma 6.3.1; Further, we have $R\cos\psi \geq A_1 - D_1$ since for a fixed r, $0 \leq r < 1$, $A_b - D_b$ decreases as b increases over the interval [0,1]. Thus, for all b, $0 \leq b \leq 1$,

$$F_b(R) = \{2[A_1(A_1-D_1)\,\frac{1-s^2r^2}{1-r^2}\,-t] + [\,D_1^2-A_1^2+(A_1-D_1)^2(\frac{1-s^2r^2}{1-r^2})]\,\}\,(A_1-D_1)$$

$$> 0.$$

Hence the minimum of $U_b(\xi,\eta)$ inside the circle $|p(z)-A_b| \leq D_b$ is attained on the diameter. Setting $\eta = 0$ in(6.3.4), we obtain

$$(6.3.7) \quad L_b(R) = (k + \frac{1-s^2r^2}{1-r^2})R + \frac{(1+t)(1-tr^2)}{(1-r^2)} R^{-1} - 2A_1(\frac{1-s^2r^2}{1-r^2})$$

where $R_b = A_b + \xi \varepsilon [A_b - D_b, A_b + D_b]$. Thus the absolute minimum of $L_b(R)$ in $(0,\infty)$ is attained at

$$(6.3.8) \qquad R' = (\frac{(1+t)(1-tr^2)}{k(1-r^2) + 1-s^2r^2})^{1/2}$$

and the value of this minimum is equal to

$$(6.3.9) \quad L_b(R') = \frac{1}{1-r^2} [\sqrt{\{k(1-r^2)+1-s^2r^2\} (1+t)(1-tr^2)} - (1-str^2)].$$

It can be easily seen that $R' < A_b + D_b$; but R' is not always greater than A_b-D_b. In such a case, when $R' \notin [A_b-D_b, A_b+D_b]$, the minimum of $L_b(R)$ on the segment $[A_b-D_b, A_b+D_b]$ is attained at $R_b = A_b-D_b$. The value of this minimum equals

$$L_b(R_b) \equiv L_b(A_b-D_b) = W/W^*,$$

where W and W^* are given by (6.3.3;a) and (6.3.3;b). Moreover $L_b(R') = L_b(R_b)$ for those values of k, s and t for which $R_b = R'$. Hence the lemma.

<u>The class</u> $R(a;\alpha,\beta)$. If $f(z) = z + a_2z^2 + \ldots$ is in $R(\alpha,\beta)$, then we have shown in Theorem 4.2.1 that $|a_2| \leq \beta(1-\alpha)$. Define

$$R(a;\alpha,\beta) = \{f(z) = z + a\beta(1-\alpha)z^2 + \ldots : f' \varepsilon P(a;\alpha,\beta), 0 \leq a \leq 1\}.$$

Goel [20] obtained the radii of convexity for the class $R(a;1-\alpha,1/2) \equiv \bar{R}(a;\alpha)$, $1/2 \leq \alpha < 1$. Recently, McCarty [52] determined exact radii of convexity for the class $R(a;\alpha,1) \equiv R(a;\alpha)$. Here we determine a sharp estimate for the radii of convexity for functions in $R(a;\alpha,\beta)$. Thus, we have the following result :

<u>Theorem 6.3.1</u> : <u>Each</u> $f \in R(a;\alpha,\beta)$ <u>maps</u> $|z| < r'$ <u>onto a convex region where</u> r' <u>is the smallest positive root of the equation</u>

$$1+4\alpha\beta ar+(4\alpha\beta^2a^2-2(1+\beta-3\alpha\beta))r^2+4\beta(2\alpha\beta-1)ar^3+(2\beta-1)(2\alpha\beta-1)r^4 = 0$$

<u>if</u> $R_a \geq R'$ <u>and</u> $r' = [\{-\alpha\beta + \sqrt{\alpha(1-2\alpha\beta+\alpha\beta^2)}\}/(1-2\alpha\beta)]^{1/2}$ <u>if</u> $R_a \leq R'$, <u>where</u>

$$R_a = \frac{1+2\alpha\beta ar+(2\alpha\beta-1)r^2}{1+2\beta ar+(2\beta-1)r^2}, \quad R' = \left(\frac{\alpha(1-(2\alpha\beta-1)r^2)}{1-(2\beta-1)r^2}\right)^{1/2}$$

<u>and</u> $r = |z| < 1$. <u>The result is sharp for each</u> α,β $(0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ <u>and</u> $0 \leq a \leq 1$.

<u>Proof</u> : Since $f \in R(a;\alpha,\beta)$, we have

$$f'(z) = \frac{1+(2\alpha\beta-1)\omega(z)}{1+(2\beta-1)\omega(z)} \tag{6.3.10}$$

where $\omega \in \mathcal{B}$. Logarithmic differentiation of (6.3.10) gives

$$1+z\frac{f''(z)}{f'(z)} = 1-2\beta(1-\alpha)\left\{\frac{z\omega'(z)}{(1+(2\beta-1)\omega(z))(1+(2\alpha\beta-1)\omega(z))}\right\}. \tag{6.3.11}$$

Applying (3.2.2) with $k = 1$, $s = 2\beta-1$, $t = 2\alpha\beta-1$ to (6.3.11), we get

$$(6.3.12)\quad \operatorname{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} \geq \frac{1}{2\beta(1-\alpha)}\left[\operatorname{Re}\left\{(2\beta-1)p(z)+\frac{2\alpha\beta-1}{p(z)}\right\} - \frac{r^2|(2\beta-1)p(z)+1-2\alpha\beta|^2-|1-p(z)|^2}{(1-r^2)\,|p(z)|}\right]+\frac{1-2\alpha\beta}{\beta(1-\alpha)}$$

where $p(z) = (1+(2\alpha\beta-1)\omega(z))/(1+(2\beta-1)\,\omega(z))$.

An application of Lemma 6.3.2 with $k = s = 2\beta - 1$, $t = 2\alpha\beta-1$ to (6.3.12) gives

$$(6.3.13)\quad \operatorname{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} \geq \begin{cases} \dfrac{1}{\beta(1-\alpha)(1-r^2)}\Big[\sqrt{4\alpha\beta^2(1-(2\beta-1)r^2)(1+(1-2\alpha\beta)r^2)} \\ \qquad -(1+(1-2\alpha\beta)(2\beta-1)r^2)+(1-2\alpha\beta)(1-r^2)\Big] \\ \qquad\qquad \text{if } R_a \leq R', \\ \dfrac{1+4\alpha\beta ar+(4\beta^2\alpha a^2-2(1+\beta-3\alpha\beta))r^2 + 4\beta(2\beta-1)(2\alpha\beta-1)ar^3+(2\beta-1)(2\alpha\beta-1)r^4}{(1+2\beta ar+(2\beta-1)r^2)(1+2\alpha\beta ar+(2\alpha\beta-1)r^2)} \\ \qquad\qquad \text{if } R_a \geq R' \end{cases}$$

where

$$R_a = \frac{1+2\alpha\beta ar+(2\alpha\beta-1)r^2}{1+2\beta ar+(2\beta-1)r^2}, \quad R' = \left(\frac{\alpha(1-(2\alpha\beta-1)\,r^2)}{(1-(2\beta-1)r^2)}\right)^{1/2}, \quad 0 \leq a \leq 1.$$

Now the theorem follows easily from (6.3.13).

The functions given by

$$f'(z) = \frac{1-2\alpha\beta az+(2\alpha\beta-1)\,z^2}{1-2\beta az+(2\beta-1)\,z^2} \quad \text{if } R_a \geq R',$$

$$f'(z) = \frac{1-2\alpha\beta cz+(2\alpha\beta-1)\,z^2}{1-2\beta cz+(2\beta-1)\,z^2} \quad \text{if } R_a \leq R'$$

where c is determined from $R_c = R'$, show that the results obtained in the theorem are sharp.

Putting $\beta = 1$, in Theorem 6.3.1, we get the following result due to McCarty [52].

Corollary 6.3.1a : Each $f \in R(a;\alpha)$ maps $|z| < r'$ onto a convex region where r' is the smallest positive root of the equation

$$1+4\alpha ar + (6\alpha-4+4\alpha a^2)r^2 + 4(2\alpha-1)ar^3+(2\alpha-1)r^4 = 0$$

if $R_a \geq R'$ and $r' = [\{-\alpha+\sqrt{\alpha(1-\alpha)}\}/(1-2\alpha)]^{1/2}$ if $R_a \leq R'$ where

$$R_a = \frac{1+2\alpha ar+(2\alpha-1)r^2}{1+2ar+r^2}, \quad R' = \left(\frac{\alpha(1-(2\alpha-1)r^2)}{1-r^2}\right)^{\frac{1}{2}}, \quad |z| = r < 1, \quad 0 \leq a \leq 1.$$

The result is sharp for each α, $0 \leq \alpha < 1$ and $0 \leq a \leq 1$.

Corollary 6.3.1b: Each $f \in \bar{R}(a;\alpha)$ maps $|z| < r'$ onto a convex region where r' is the smallest positive root of the equation

$$1 + 2(1-\alpha)ar + ((1-\alpha)a^2-3\alpha)\, r^2-2\alpha ar^3 = 0$$

if $R_a \geq R'$ and $r' = [\{-(1-\alpha)+\sqrt{(1-\alpha)(1+3\alpha)}\}/2\alpha]^{1/2}$ if $R_a \leq R'$, where

$$R_a = \frac{1+(1-\alpha)ar - \alpha r^2}{1+ar}, \quad R' = [(1-\alpha)(1+\alpha r^2)]^{1/2} \text{ and } r = |z| < 1.$$

The result is sharp for each α $(0 \leq \alpha < 1)$ and $0 \leq a \leq 1$.

This result is obtained by replacing α by $1-\alpha$ and β by $1/2$ in Theorem 6.3.1. It may be noted that this result was obtained by Goel [20] under the additional restriction $1/2 \leq \alpha \leq 1$.

Corollary 6.3.1c: If $f(z) = z + a\gamma z^2 + \ldots$ such that $f \in R(\gamma)$, then f maps $|z| < r'$ onto a convex region where r' is the smallest positive root of the equation

$$1 + 2a(1-\gamma)r+(a^2(1-\gamma^2) - 4\gamma)r^2-2a\gamma(1+\gamma)r^3 - \gamma^2r^4 = 0$$

<u>if</u> $R_a \geq R'$ <u>and</u> $r' = [\{(\gamma^2-1) + \sqrt{(1-\gamma^2)(1+4\gamma-4\gamma^2)}\}/2\gamma(1+\gamma)]^{1/2}$ <u>if</u> $R_a \leq R'$, <u>where</u>

$$R_a = \frac{1+(1-\gamma)ar - \gamma r^2}{1+(1+\gamma)ar + \gamma r^2}, \quad R' = \left(\frac{(1-\gamma)(1+\gamma r^2)}{(1+\gamma)(1-\gamma r^2)}\right)^{1/2}, \quad r = |z| < 1,$$

$$0 \leq a \leq 1.$$

<u>The result is sharp for each</u> γ, $0 < \gamma \leq 1$ <u>and</u> $0 \leq a \leq 1$.

This result is obtained by replacing α by $(1-\gamma)/(1+\gamma)$ and β by $(1+\gamma)/2$ in Theorem 6.3.1.

<u>Remark 6.3.1</u> : The cases $(\alpha,\beta) = (0,1-\delta)$ and $(\alpha,\beta) = (0,(2\delta-1)/2\delta)$, $\delta > 1/2$ in Theorem 6.3.1 lead to the corresponding results for the functions of the classes introduced by Shaffer [76] and Goel [20] respectively.

<u>The Class</u> $S^*(a;\alpha,\beta)$. If $f(z) = z + a_2z^2 + \ldots$ is in $S^*(\alpha,\beta)$, then we have shown in Theorem 2.5.1 that $|a_2| \leq 2\beta(1-\alpha)$. Define

$$S^*(a;\alpha,\beta) = \{g(z)=z+2a\beta(1-\alpha)z^2+\ldots : zg'/g \in P(a;\alpha,\beta),\ 0 \leq a \leq 1\}.$$

Tepper [81] found the radius of convexity for functions in $S^*(a;0,1) = \bar{S}^{**}_a$. Recently McCarty [52] has obtained the radii of convexity for functions in $S^*(a;\alpha,1) \equiv S^*(a;\alpha)$. Here we determine sharp estimates for the radii of convexity for functions in $S^*(a;\alpha,\beta)$. Thus we have the following :

<u>Theorem 6.3.2</u> : <u>Each</u> $g \in S^*(a;\alpha,\beta)$ <u>maps</u> $|z| < r'$ <u>onto a convex region where</u> r' <u>is the smallest positive root of the equation</u>

$$1+2(3\alpha-1)\beta ar+(4\alpha^2\beta^2a^2+8\alpha\beta-2-4\beta)r^2-2\beta(1+\alpha-4\alpha^2\beta)ar^3+(1-2\alpha\beta)^2\ r^4 = 0$$

if $R_a \geq R'$ and $r' = [\,(5\alpha-1)/\{(1-\alpha+4\alpha^2\beta)+4\alpha\sqrt{(1+\beta-3\alpha\beta+\alpha^2\beta^2)}\}]^{1/2}$ if $R_a \leq R'$, where

$$R_a = \frac{1 + 2\alpha\beta ar + (2\alpha\beta-1)r^2}{1 + 2\beta ar + (2\beta-1)\ r^2}, \quad R' = \left(\frac{\alpha(1 + (1-2\alpha\beta)\ r^2)}{(2-\alpha) - (2\beta-\alpha)\ r^2}\right)^{1/2} \text{ and}$$

$$|z| = r < 1.$$

The result is sharp for each α, β $(0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ and $0 \leq a \leq 1$.

Proof : Since $f \in S^*(a;\alpha,\beta)$, we have

$$z\,\frac{g'(z)}{g(z)} = \frac{1 + (2\alpha\beta-1)\omega(z)}{1 + (2\beta-1)\ \omega(z)} \tag{6.3.14}$$

where $\omega \in \mathcal{B}$. Differentiating (6.3.14) logarithmically, we have

$$1+z\,\frac{g''(z)}{g'(z)} = \frac{1+(2\alpha\beta-1)\omega(z)}{1+(2\beta-1)\ \omega(z)} - 2\beta(1-\alpha)\left\{\frac{z\omega'(z)}{(1+(2\beta-1)\omega(z))(1+(2\alpha\beta-1)\omega(z))}\right\}. \tag{6.3.15}$$

Applying (3.2.2) with $k = 1$, $s = 2\beta-1$, $t = 2\alpha\beta-1$ to (6.3.15), we get

$$\operatorname{Re}\left\{1 + z\,\frac{g''(z)}{g'(z)}\right\} \geq \frac{1}{2\beta(1-\alpha)}\left[\operatorname{Re}\left\{(4\beta-1-2\alpha\beta)p(z) + \frac{2\alpha\beta-1}{p(z)}\right\} - \frac{r^2|(2\beta-1)p(z)+1-2\alpha\beta|^2-|1-p(z)|^2}{(1-r^2)\ |p(z)|}\right] + \frac{\alpha(1+\beta)-1}{\beta(1-\alpha)}, \tag{6.3.16}$$

where $p(z) = (1+(2\alpha\beta-1)\omega(z))/(1+(2\beta-1)\omega(z))$.

Now, an application of Lemma 6.3.2 with $k = 4\beta-1-2\alpha\beta$, $s = 2\beta-1$ and $t = 2\alpha\beta-1$ to (6.3.16), gives the required results easily.

The functions given by

$$z\frac{g'(z)}{g(z)} = \frac{1-2\alpha\beta az + (2\alpha\beta-1)\, z^2}{1-2\beta az + (2\beta-1)\, z^2} \quad \text{if } R_a \geq R'$$

and

$$z\frac{g'(z)}{g(z)} = \frac{1-2\alpha\beta cz + (2\alpha\beta-1)\, z^2}{1-2\beta\, cz + (2\beta-1)\, z^2} \quad \text{if } R_a \leq R'$$

where c is determined by the relation $R_c = R'$ show that the results obtained in the theorem are sharp.

Putting $\beta = 1$, in Theorem 6.3.2 gives the following result obtained by McCarty [52] .

Corollary 6.3.2a: Each $f \in S^*(a;\alpha)$ maps $|z| < r'$ onto a convex region where r' is the smallest positive root of the equation

$$1+(6\alpha-2)ar+(4\alpha^2a^2+8\alpha-6)r^2+(8\alpha^2-2\alpha-2)ar^3+(2\alpha-1)^2r^4 = 0$$

if $R_a \geq R'$ and $r' = [(5\alpha-1)/\{(4\alpha^2-\alpha+1)+4\alpha\sqrt{(\alpha^2-3\alpha+2)}\}]^{1/2}$ if $R_a \leq R'$ where

$$R_a = \frac{1+2\alpha ar + (2\alpha-1)\, r^2}{1+2ar + r^2}, \quad R' = \left(\frac{\alpha(1-(2\alpha-1)\, r^2)}{(2-\alpha)(1-r^2)}\right)^{1/2}, |z|=r<1, 0 \leq a \leq 1.$$

The result is sharp for each α, $0 \leq \alpha < 1$ and $0 \leq a \leq 1$.

Corollary 6.3.2b : If $f(z) = z + 2a\gamma z^2 + \ldots$ such that $f \in S^*(\gamma)$, then f maps $|z| < r'$ onto a convex region where r' is the smallest positive root of the equation

$$1+2a(1-2\gamma)r+(a^2(1-\gamma)^2-6\gamma)r^2+2a(\gamma^2-2\gamma)r^3 + \gamma^2r^4 = 0$$

if $R_a \geq R'$ and $r' = [\{-(1-\gamma+\gamma^2)+(1-\gamma)\sqrt{(\gamma^2+6\gamma+1)}\}/(3\gamma-\gamma^2)]^{1/2}$ if $R_a \leq R'$

where

$$R_a = \frac{1 + (1-\gamma)\,ar - \gamma r^2}{1 + (1+\gamma)\,ar + \gamma r^2}, \quad R' = \left(\frac{(1-\gamma)(1+\gamma r^2)}{2\gamma(1-r^2)+1-\gamma^2 r^{2k}}\right)^{\frac{1}{2}}, \; |z|=r < 1, 0 \leq a \leq 1.$$

The results are sharp for each $\gamma (0 < \gamma \leq 1)$ and $0 \leq a \leq 1$.

This result is obtained by replacing α by $(1-\gamma)/(1+\gamma)$ and β by $(1+\gamma)/2$ in Theorem 6.3.2.

Remark 6.3.2 : (a) The cases $(\alpha, \beta) = (0, 1/2)$; $(\alpha, \beta) = (0, (2\delta-1)/2\delta)$, $\delta > 1/2$ and replacement of α by $1-\alpha$ and β by $1/2$, in Theorem 6.3.2 lead to analogous results for the functions of the classes introduced and studied by Eenigenburg [13], Ram Singh [69], Wright [83] etc.

(b) $(\alpha, \beta) = (0, 1)$ in Theorem 6.3.2 gives the corresponding result due to Tepper [81].

6.4 The radii of starlikeness for functions in $V^*(a; \alpha, \beta)$.

In Theorem 3.4.1, we have obtained the radii of starlikeness for functions in $V^*(\alpha, \beta)$. Let $V^*(a; \alpha, \beta)$, be the class of functions of the form

$$(6.4.1) \qquad F(z) = 1/2(zf(z))'$$

where $f \in S^*(a; \alpha, \beta)$. Recently, Al-Amiri [1] has determined the radii of starlikeness for the class $V^*(a, 0, 1) \equiv \bar{V}^*_a$. Here we obtain exact bounds for the radii of starlikeness for functions in $V^*(a; \alpha, \beta)$. Thus we have the following result :

Theorem 6.4.1 : Each $F \in V^*(a; \alpha, \beta)$ maps $|z| < r'$ onto a starlike region where r' is the smallest positive root of the equation

$$1+4\alpha\beta ar+(2\alpha(1+\alpha)\beta^2a^2-2-\beta+5\alpha\beta)r^2+2a(1-\alpha+2\alpha\beta)(2\alpha\beta-1)ar^3+\alpha(2\beta-1)(2\alpha\beta-1)\ r^4 = 0$$

<u>if</u> $R_a \geq R'$ <u>and</u> $r=[4\alpha/\{2\alpha\beta(1+\alpha)+\sqrt{(4\beta^2\alpha^4-4\beta(4-2\beta)\alpha^3+4(2-4\beta+\beta^2)\alpha^2+8\alpha}\}]^{1/2}$

<u>if</u> $R_a \leq R'$, <u>where</u>

$$R_a = \frac{1+(1+\alpha)\beta ar + (\beta(1+\alpha)-1)r^2}{1+2\beta ar + (2\beta-1)r^2}, \quad R' = \left(\frac{(1+\alpha)(1-(\alpha\beta+\beta-1)r^2)}{2\,[(2-\alpha)-(2\beta-\alpha)r^2]}\right)^{1/2}$$

$r = |z| < 1$, $0 \leq a \leq 1$. <u>The result is sharp for each</u> α, $\beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$ <u>and</u> $0 \leq a \leq 1$.

<u>Proof</u> : Since $f \in S^*(a;\alpha,\beta)$, we have

$$z\frac{f'(z)}{f(z)} = \frac{1 + (2\alpha\beta-1)\omega(z)}{1 + (2\beta-1)\ \omega(z)} \tag{6.4.2}$$

where $\omega \in B$. Differentiating (6.4.1) and using (6.4.2), we obtain

$$z\frac{F'(z)}{F(z)} = \frac{1+(2\alpha\beta-1)\omega(z)}{1+(2\beta-1)\omega(z)} - (1-\alpha)\beta\left\{\frac{z\omega'(z)}{(1+(2\beta-1)\omega(z))(1+(\alpha\beta+\beta-1)\omega(z))}\right\}. \tag{6.4.3}$$

Applying (3.2.2) with $k = 1$, $s = 2\beta-1$, $t = \alpha\beta+\beta-1$ to (6.4.3), we get

$$\operatorname{Re}\left\{z\frac{F'(z)}{F(z)}\right\} \geq \frac{1}{\beta(1-\alpha)}\left[\operatorname{Re}\left\{(4\beta-1-2\alpha\beta)p(z) + \frac{\alpha\beta+\beta-1}{p(z)}\right\} - \frac{r^2|(2\beta-1)p(z)-(\alpha\beta+\beta-1)|^2-|1-p(z)|^2}{(1-r^2)\ |p(z)|}\right] - \frac{(2\beta-1)}{\beta(1-\alpha)} \tag{6.4.4}$$

where $p(z) = (1+(\alpha\beta+\beta-1)\ \omega(z))/(1+(2\beta-1)\ \omega(z))$.

Again the result follows easily, by applying Lemma 6.3.2 with $k = 4\beta-1-2\alpha\beta$, $s = 2\beta-1$, $t = \alpha\beta+\beta-1$ to (6.4.4).

The functions given by

$$z\frac{f'(z)}{f(z)} = \frac{1+\beta(1+\alpha)az+(\alpha\beta+\beta-1)z^2}{1-2\beta az+(2\beta-1)z^2} \text{ if } R_a \geq R'$$

and

$$z\frac{f'(z)}{f(z)} = \frac{1+\beta(1+\alpha)cz+(\alpha\beta+\beta-1)z^2}{1-2\beta cz+(2\beta-1)z^2} \text{ if } R_a \leq R'$$

where c is determined by the relation $R_c = R'$ show that the results obtained in the theorem are sharp.

$(\alpha,\beta) = (0,1)$ in Theorem 6.4.1, gives the following result for functions in $\bar{V}^*_a$ obtained by Al-Amiri [1] .

Corollary 6.4.1 : If $F(z) = 1/2(zf(z))'$, where $f(z) = z + 2az^2 + a_3z^3 + \ldots$ is in S^*, then F is starlike for $|z| < r_0$, where r_0 is the smallest positive root of the equation $1 - 3r^2 - 2ar^3 = 0$.

Remark 6.4.1 : Putting different values of the parameters α, β $(0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ in the above theorem, we can easily derive analogous results depending on second coefficient for the classes of functions of the form (6.4.1) where f belongs to the different subclasses of starlike functions defined in section 1.3.

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